CONTENTS

INSTRUCTIONS TO CONTRIBUTORS

Articles should be as *brief* as possible. The viewpoint should be comprehensive in giving the relation of the paper to previous publications of the author or of others and in exhibiting, when practicable, the significance of the work for other branches of science. Elaborate technical details of the work and long tables of data should be avoided, but authors should be precise in making clear the new results and should give some record of the methods and data upon which they are based.

Manuscripts should be prepared with a current number of the PROCEEDINGS as a model in matters of form, and should be type-written in duplicate with double spacing, the author retaining one copy. Illustrations should be confined to text-figures of simple character, though more elaborate illustrations may be allowed in special instances to authors willing to pay for their preparation and insertion. Authors are requested to send in all drawings or other illustrations in a state suitable for direct photographic reproduction. They should be drawn on a large scale in Indian Ink on Bristol board, with temporary lettering in pencil. Great care should be exercised in selecting only those that are essential. If unsatisfactory drawings are submitted, authors may be required to have them redrawn by a professional artist. Particular attention should be given to arranging tabular matter in a simple and concise manner.

References to literature will be placed at the end of the article and short footnotes should be avoided. It is suggested that references to periodicals be furnished in some details and in general in accordance with the standard adopted for the Subject Catalogue of the International Catalogue of Scientific Literature, *viz.*, name of author, arranged alphabetically, with initials following (ordinarily omitting title of paper), year, abbreviated name of Journal, volume, inclusive pages.

Papers by members of the Academy may be sent to the General Secretary, National Academy of Sciences, Allahabad. Papers by non-members of the Academy should be submitted through a member.

No Papers exceeding seven pages of printed matter will be published in the PROCEEDINGS except with a special permission of the Council.

Every paper must be accompanied by *three copies* of a brief summary not exceeding 300 words in length.

A Proof will ordinarily be sent and the author is requested to return it with the manuscript, at his earliest convenience. All proof corrections involve heavy expenses which would be negligible if the papers are carefully revised by the authors before submission.

THE CHEMICAL EXAMINATION OF *SPHAERANTHUS INDICUS* LINN. PART II

The Constituents

BY RAM DAS TIWARI

CHEMISTRY DEPT., UNIVERSITY OF ALLAHABAD

(*Received on August* 13, 1946).

SUMMARY

A glucoside having the molecular formula $C_{22}H_{26}O_{12}$ has been isolated from *Sphaeranthus indicus*. A higher alocohol $C_{30}H_{62}O$ has also been found in the unsaponifiable matter of the fatty oil of the material.

In a previous communication[1], the author examined the component fatty acids and the probable glyceride structure of the fatty oil from *Sphaeranthus indicus*. In the present communication, other constituents of the plant have been isolated and studied. A preliminary examination of the material has shown the presence of albumins, tannins, reducing sugars, fatty oil, a volatile oil and some glucoside but no alkaloid could be detected. This, however, is contrary to the report of Dymock[2].

For detailed examination the material was extracted with rectified spirit and from the alcoholic extract various constituents were isolated and examined as described in the experimental part. The constants of the volatile oil have been determined and found to be in fair agreement with those obtained by Sanjiva Rao and co-workers[3]. The component fatty acids and the probable glyceride structure of the fatty oil have already been reported. Also from the unsaponifiable matter of the fatty oil a compound M. P. 78°C was isolated which was identified to be a higher aliphatic

alcohol $C_{30} H_{62} O$. In spite of the analytical results of this compound, nothing definite can be said regarding its exact composition because it has been definitely shown by Chibnall[4] et. al. that the alcohols of higher molecular weight are mixtures of alcohols with even number of carbon atoms from C_{24} to C_{36}. Usually a complex mixture of these compounds, inseparable by methods at present available, is present.

The chief constituent of the material is a glucoside $C_{22} H_{26} C_{12}$ M.P. 148-149°C. It is slightly soluble in cold alcohol and more on warming and gives no colour with ferric chloride. It dissolves in concentrated sulphuric acid with red colour and in concentrated nitric acid with yellow colour which turns to red on heating. On boiling with dilute sulphuric acid it gave glucose and another compound which was soluble in water and did not separate out from the product of hydrolysis. The aglucone appears to be a phenolic compound as a blue green colour is obtained with ferric chloride.

Besides the above glucoside, potassium chloride, tannins and reducing sugars chiefly glucose have also been identified. The exact constitution of the glucoside will be studied and reported afterwards.

EXPERIMENTAL.

The material employed for the investigation consisted of the capitulas of *Sphaeranthus indicus* obtained from the Punjab Ayurvedic Pharmacy, Amritsar and identified locally.

Thirty grams of the crushed material was incinerated completely on a porcelain dish when 2·91 gms. (9·7 per cent.) of a dirty white ash was obtained, which on examination was found to contain the following basic and acidic radicals :—Aluminium, calcium, magnesium, sodium, potassium, chloride, carbonate and sulphate.

About twenty gms. of the crushed material was extracted with boiling water. The extract gave blue colour with ferric chloride, a precipitate with lead acetate solution and reduced Fehling solution. No test for alkaloids was obtained. The presence of tannins and reducing sugars is thus indicated.

A small amount of the crushed material was then extracted with one per cent hydrochloric acid solution but the extract gave no test for alkaloid with the alkaloidal reagents. About twenty grams of the crushed

material was mixed with moist lime and distilled. No tests for alkaloid were obtained either in the distillate or in the liquid left in the flask. The plant, therefore, does not appear to contain any alkaloid.

The average weight of a capitula was 0·1115 gms. and on being crushed they were found to contain 3·91 per cent of moisture. In order to have an approximate idea regarding the solubility of the constituents, twentyfive grams of the powdered material was extracted in a soxhelet apparatus with a number of solvents in the given order and the following results were obtained :—

(1) *Petroleum ether* (40-60°) a dark coloured extract was obtained which on distillation of the solvent gave a dark green oil having a good characteristic odour yield 5·03 per cent. This consisted mostly of the fat.

(2) *Absolute ether.*—a thick dark resinous mass was obtained. Yield 1·1 per cent. This gave no test for alkaloid or glucoside and consisted mostly of the chlorophyll.

(3) *Chloroform.*—a dark grey mass consisting mostly of the chlorophyll was obtained. Yield 0·9 per cent.

(4) *Ethyl acetate.*—A light blue green extract was obtained which gave a dirty green mass consisting of tannins and chlorophyll. Yield 0·7 per cent.

(5) *Absolute alcohol.*—A dark brown extract was obtained from which a dirty brown mass was obtained on distilling off the solvent. The residue consisted of tannins and glucoside with some chlorophyll. Yield 14.3 per cent.

(6) *Alcohol 70 per cent.*—An yellow brown extract which gave test for tannins and reducing sugars was obtained. Yield 7·8 per cent.

The presence of albumins was also indicated by extracting the residue with caustic soda. A part of the material was also distilled with water when a distillate having pleasant characteristic smell was obtained, showing the presence of essential oil in the material.

The crushed material was so light and voluminous that hardly 600 gms. could be put in a five litre extraction flask and when about 4 litres solvent was added, only about a litre could be filtered back, the rest of it was absorbed by the material. It became impossible, therefore, to use petroleum ether or benzene for extraction of the fatty oil from the material, more so because of the low yield of the oil (5 per cent).

Hence for the purpose of complete examination about twenty kilogram-mes of the crushed material was extracted with rectified spirit in lots of 600 gms. in a five litre extraction flask. The whole extraction took about eight months.

The alcoholic extract on concentration and keeping deposited 3·2 gms. of a residue (R). On distilling off most of the solvent a syrupy mass having a characteristic odour was left. This was distilled in a current of steam. The steam distillate was extracted with ether when 14·86 gms. of a volatile oil (V) having a characteritstic smell was obtained. Now there remained in the distillation flask (1) a black solid resinous mass (D) at the bottom (2) an aqueous layer (L) over the black resinous mass and (3) an oily layer floating over the aqueous layer.

Examination of the residue (R).—This melted between 80-120°C. and appeared to be partly inorganic. On ignition it left 2·29 per cent of ash. On repeated crystallisations from boiling absolute alcohol 1·1 gms. of a light yellow substance melting point 85-86°C was obtained. This was free from any inorganic impurities, was insoluble in cold or hot water, dilute acids, dilute or strong alkali but was soluble in concentrated acids on warming, giving a red colour. The substance was however not of distinct crystalline character and was therefore not analysed.

Examination of the volatile oil (V).—This was viscous light orange brown coloured oil having a characteristic odour and bitter taste. It was soluble in water and alcohol. The physical and chemical constants of the oil are given below. The constants of the oil as reported by Sanjiva Rao *et. al.* (*loc. cit.*) are also given for comparison.

Constants.	Results of the author	Results of Sanjiva Rao *et. al.*
Yield on the material	0·07 per cent	0·27 to 0·37
Specific gravity	0·9679 (30°C)	0·9675 to 0·9684 (30°C)
Refractive index	1·5034(25°C)	1·5032 to 1·504 (30°C)
Acid value	4·9	1·4 to 3·2
Saponification value.	20·6	4·8 to 14·4
Ester value	15·7	
Saponification value after acetylation	112·3	117·6 to 136·2

The detailed examination of the constituents of the oil will be done when about ten maunds of the fresh material will be available. As the most common way of the use of the drug is in the form of its stream distillate, there is no wonder that the active principle of the plant may be some constituent of the volatile oil. The study of this will therefore be very interesting.

Examination of the black solid resinous mass (D).—The black resinous mass weighing 12 gms. (0·06 %.) was insoluble in water. It was dried for about fifteen days in a vacuum dessicator and mixed with 30 gms. of the powdered pumice stone and powdered finally. Then this powdered mass was extracted successively with ether, chloroform ethyl acetate and absolute alcohol in a soxhelet extractor, but from none of these could a crystalline compound be isolated. In all cases a resinous mass was obtained. Hence this part could not be examined in detail.

Examination of the aqueous layer (L).—The aqueous layer was first extracted with ether, chloroform and amyl alcohol in succession. The ether and chloroform extracts contained small amounts of oily resinous mass and the amyl alcohol extract contained mostly tannins. This was then concentrated to a small volume and kept for about a week inside a frigidaire when it deposited a residue which was of inorganic nature. No more of this residue was obtained on further concentration. Thus 10·3 gms. of this residue was obtained (yield 0·05 % on the material). This was purified by crystallising from water and was identified to be potassium chloride.

To the liquid after separation of the above residue, a solution of lead acetate was added, when an yellow precipitate was obtained. The precipitate was filtered off, washed, suspended in water and decomposed with hydrogen sulphide. The lead sulphide was filtered off and hydrogen sulphide was driven off from the solution by passing carbon dioxide and heating. This was then concentrated and kept but no crystalline compound separated. The water was then completely removed when only tannins were left.

To the main filterate a solution of basic lead acetate was added when a small amount of precipitate was obtained. This was filtered off and

decomposed with hydrogen sulphide as above, but only small amounts of tannins were obtained.

The mother liquor after filtering off the precipitate with basic lead acetate was treated with hydrogen sulphide to remove excess of lead. The solution was then freed from hydrogen sulphide by bubbling in carbon dioxide and heating, concentrated and kept but no solid separated even on keeping it for ten days inside a frigidaire. This was a syrupy mass smelling strongly of reducing sugars. On treatment with phenylhydrazine and glacial acetic acid an osazone was obtained which after crystallisation from 70%. alcohol was found to melt at 205°C. This melting point was not depressed by the addition of a pure sample of glucosazone. The reducing sugars therefore chiefly consist of glucose.

The aqueous layer has thus been found to consist of potassium chloride, tannins and reducing sugars.

The examination of the oily layer.—The oily layer was separated from the aqueous layer by means of a separating funnel and was then extracted with petroleum ether (B. P. 40-60°C). The extract was washed, dehydrated and the solvent was distilled off when 682 gms. of a fatty oil having a dark green colour was obtained (yield 3·4 per cent. on the material). This was a semi drying oil the examination of the component fatty acids and probable glyceride structure of which has already been reported in part I.

While describing the results of the examination of the unsaponifiable matter of the fatty oil, it was stated that the unsaponifiable matter consists of a compound melting point 78°C, besides a phytosterol M. P. 135°C. This compound on repeated crystallisation from absolute alcohol melted at 79·5°C. This was found to be a primary aliphatic alcohol. On analysis the compound was found to contain C=82·49 per cent. and H= 13·83 per cent; $C_{30} H_{61}$ (OH) requires C=82·19 per cent. and H= 14.16. per cent. Also found molecular weight [Rast] 439 and $C_{30} H_{61}$ (CH) requires 438.

On boiling with acetic anhydride and sodium acetate it gave an acetyl derivative which after crystallisation from alcohol melted at 68-69°C. Benzoyl derivative was similarly prepared and crystallised twice from absolute alcohol when it melted at 70°C.

In spite of these analytical results, nothing definite can be said regarding the exact composition of this alcohol because it has been definitely shown by Chibnall et. al. (loc. cit) that alcohol of higher molecular weight are mixtures of alcohol with even number of carbon atoms from C_{24} to C_{36}. Usually a somewhat complex mixture of these compounds inseparable by methods at present available is presented. Hence no reliance can be placed on analytical data.

After removing the fatty oil from the oily layer with petorleum ether (B. P. 40-60°C), the remaining portion was extracted with ether when 4·6 gms. of a resinous mass was obtained. Attempts to crystallise it from various solvents were not successful.

The other insoluble portion of above was fractionally crystallised from absolute alcohol whereby two substances were obtained. One was a dirty white substance M.P. 145°C and the other was a light yellow amorphous mass with no sharp melting point and appeared to be partly inorganic.

Isolation of the glucoside.—The dirty white substance M.P. 145°C as obtained above was repeatedly crystallised from alcohol when white silky needles M.P. 148-149°C were obtained. Further crystallisation did not raise the melting point. 3·5 gms of the substance was thus obtained (yield 0·002 per cent.). The substance does not contain nitrogen, sulphur or halogen and is aromatic in character—It is insoluble in cold or hot water, slightly soluble in cold alcohol more on warming, insoluble in dilute or strong alkali and cold dilute acids. It dissolves in hot concentrated sulphuric acid with red colour and in concentrated nitric acid with yellow colour which turns red on heating. The substance does not give any colour with ferric chloride in alcoholic solution nor does it reduce Fehling Solution. On analysis it was found to contain C = 54·64 per cent. and H = 5·12 per cent; $C_{22} H_{26} O_{12}$ requires C = 54·78 per cent and H = 5·40 per cent. Also found molecular weight 490 [Rast] $C_{22} H_{26} O_{12}$ requires 482.

On boiling for two hours with dilute sulphuric acid, it reduced Fehling Solution showing that it is glucosidal in character. In the solution resulting on hydrolysis glucose was detected (osazone M. P. 204°C) showing that the sugar part of the glucoside is glucose. The aglucone does not

separate on keeping the hydrolysis product showing thereby that it is water soluble. It, however, appears to be phenolic in character as a blue green colour was obtained with ferric chloride.

REFERENCES

1. Singh and Tewari : Proc. Nat. Acad. Sci. Ind., 1943, **13**, 88.
2. Dymock Warden and Hooper : Pharmacographica Indica, Vol. 2, P. 257.
3. Sanjiva Rao. *et. al.* : Perfumery and Essential Oil records, 1937, **28**, 411-13.
4. Chibnall *et. al.* : Biochem.   J., 1934. **28**. 2189.

ON THE STRONG SUMMABILITY OF THE DERIVED SERIES OF A FOURIER SERIES

By U. N. SINGH

UNIVERSITY OF ALLAHABAD

1.1   A series

$$u_0 + u_1 + u_2 + \ldots\ldots\ldots\ldots$$

is said to be strongly summable (C, 1) to the sum S, if

(1.11) $$\sum_{\nu=0}^{n} |\, S_\nu - S\,| = o\,(n)$$

where

$$S_\nu = u_0 + u_1 + \ldots\ldots\ldots + u_\nu.$$

The series is said to be summable H_k, $k > o$ if there exists a finite number S, such that

(1.12) $$\sum_{\nu=0}^{n} |\, S_\nu - S\,|^{\,k} = o\,(n)$$

1.2.   It is known[1] that the derived series of a Fourier series corresponding to a function $f\,(x)$, periodic and integrable (L), is summable (C, δ), $\delta > 1$ to the sum $\lim_{h \to 0} \dfrac{f\,(x+h) - f\,(x-h)}{2h}$ wherever this limit exists. When $f\,(x)$, in addition, is a function of bounded variation, the derived series is summable (C, 1) almost everywhere to the first differential coefficient of $f\,(x)$.[2]   With regard to the summability H_2 Marcinkiewicz[3] has proved that a Fourier series and its conjugate series are summable H_2 almost everywhere.   But when we come to consider the H_2 summability of the derived series of the Fourier series corresponding even to a function of bounded variation, the result is rather unexpected.   In fact the analogy in the results of the Fourier series and its derived series with regard to the C-summability process breaks down when the summability H_2 is applied to these series.

---

[1] Zygmund [3]   [2] Young [2]   [3] Marcinkiewicz [1]

F. 2

1.3.   In this paper we shall prove by considering an example that the derived series of a Fourier series associated with a function of bounded variation is not necessarily strongly summable (C, 1) at a point where the first differential coefficient exists.   Indeed, our example shows that the derived series may not be summable H_2 anywhere in the interval $(0, 2\pi)$.

I am much indebted to Dr. B. N. Prasad for his kind interest and advice in the preparation of this paper.

2.   Let

$$f(x) = \tfrac{1}{2}(\pi - x) \qquad (0 < x < 2\pi)$$
$$f(0) = 0$$

and we define $f(x)$ outside this interval by periodicity with period 2π. $f(x)$ is an odd function of bounded variation with just one jump at $x = 0$ in the whole period.   Denoting the Fourier constants of $f(x)$ by a_n, b_n, we obtain

$$a_n = 0 \quad \text{for every value of } n$$

and for $n > 0$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\,\sin n\theta\,d\theta = \frac{1}{2\pi}\int_0^{2\pi}(\pi - \theta)\sin n\theta\,d\theta$$

$$= \frac{1}{\pi}\int_0^{\pi}(\pi - \theta)\sin n\theta\,d\theta$$

$$= \frac{1}{\pi}\left[-\frac{(\pi - \theta)\cos n\theta}{n}\right]_0^{\pi} - \frac{1}{\pi}\int_0^{\pi}\frac{\cos n\theta}{n}\,d\theta$$

$$= \frac{1}{n}$$

Hence the Fourier series corresponding to $f(x)$ is

(2·1) $$\sum_{n=1}^{\infty} \frac{\sin nx}{n}$$

and its first derived series will be

(2·2) $$\sum_{n=1}^{\infty} \cos nx$$

Now the series (2.2) is summable (C, 1) to $-\frac{1}{2}$ i.e., to the first differential coefficient of $f(x)$, everywhere in the open interval $(0, 2\pi)$, as can easily be verified. If we denote by $S_n = S_n(x)$ the sum of the first n terms of the series (2.2), then the result we have just stated can be written as

$$\sum_{\nu=1}^{n} \{S\nu - f'(x)\} = 0 \ (n)$$

for every value of x in the open interval $(0, 2\pi)$

Here we shall prove that

(2.3)
$$\sum_{\nu=1}^{n} |\, S\nu - f'(x)\,| \ \ne o\,(n) \text{ at } x = \pi$$

(2.4)
$$\sum_{\nu=1}^{n} \frac{|\, S\nu - f'(x)\,|}{\nu} \ne o\,(\log n) \text{ at } x = \pi$$

and

(2.5)
$$\sum_{\nu=1}^{n} \{S\nu - f'(x)\}^2 \ \ne o\,(n) \text{ any where in } (0, 2\pi)$$

For, when x is not zero

$$S\nu(x) - f'(x) = \frac{1}{2} + \sum_{m=1}^{\nu} \cos mx$$

$$= \frac{\sin(\nu + \frac{1}{2})x}{2\sin\frac{x}{2}}$$

Hence for $x = \pi$

$$|\, S\nu - f'(x)\,| = \frac{|\, \sin(\nu + \frac{1}{2})\pi\,|}{2\sin\frac{\pi}{2}}$$

$$= \frac{1}{2} \text{ for all values of } \nu$$

Therefore

$$\sum_{\nu=1}^{n} |\, S\nu - f'(x)\,| = \frac{n}{2}$$

which proves (2.3)

Again,

$$\sum_{\nu=1}^{n} \frac{|\, S\nu - f'(x)\,|}{\nu} = \tfrac{1}{2} \sum_{1}^{n} \frac{1}{\nu}$$

$$= \tfrac{1}{2} \left[1 + \tfrac{1}{2} + \ldots + \frac{1}{n} - \log n \right] + \tfrac{1}{2} \log n$$

and this proves (2·4)

To prove (2.5) we observe that

$$\{S\nu - f'(x)\}^2 = \frac{\sin^2 (\nu + \tfrac{1}{2}) x}{4 \sin^2 \dfrac{x}{2}}$$

$$= \tfrac{1}{8} \frac{1 - \cos (2\nu + 1)\, x}{\sin^2 \dfrac{x}{2}}$$

$$\therefore \quad \sum_{1}^{n} \{S\nu - f'(x)\}^2 = \frac{1}{8 \sin^2 \dfrac{x}{2}} \sum_{1}^{n} \{1 - \cos (2\nu + 1)\, x\}$$

$$= \frac{1}{8 \sin^2 \dfrac{x}{2}} \left[n - \left\{ \frac{\sin 2 (n+1)x - \sin 2x}{2 \sin x} \right\} \right]$$

Hence

$$\lim_{n \to \infty} \frac{1}{n} \sum_{1}^{n} \{S\nu - f'(x)\}^2 = \frac{1}{8 \sin^2 \dfrac{x}{2}} \quad \text{in the open interval } (0, 2\pi)$$

This proves (2.5).

3.   Let $f(t)$ be a function of bounded variation, periodic and integrable (L).   Let the Fourier series corresponding to $f(t)$ be

(3.1) $$\tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)$$

Denoting by $S_n(x)$ the sum of the first n terms of the first derived series of (3.1) at the point $t = x$ we have

$$S_n(x) = \tfrac{1}{2} \pi \int_{0}^{2\pi} \frac{d}{dx} \frac{\sin (n + \tfrac{1}{2}) (x - u)}{\sin \dfrac{x - u}{2}} f(u)\, du$$

$$= -\frac{1}{2\pi}\int_0^{2\pi} f(u)\left\{\frac{d}{du}\cdot\frac{\sin\left(n+\frac{1}{2}\right)(x-u)}{\sin\frac{1}{2}(x-u)}\right\}du$$

$$= -\frac{1}{2\pi}\int_0^{\pi}\{f(x+t)-f(x-t)\}\left\{\frac{d}{dt}\frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\frac{1}{2}t}\right\}dt$$

Now integrating by parts the right-hand side we obtain

$$S_n(x) = \frac{1}{2\pi}\int_0^{\pi}\frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\frac{1}{2}t}\,d\{f(x+t)-f(x-t)\}$$

(3.2)
$$= \frac{1}{2\pi}\int_0^{\pi}\frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\frac{1}{2}t}\cdot d\phi(t) + f'(x)$$

Where $\phi(t) = \phi(t, x) = f(x+t) - f(x-t) - 2t\,f'(x)$ and $f'(x)$ denotes the first differential coefficient of $f(t)$ at the point $t=x$. We shall now prove the following theorem.

Theorem A. If at a point x where $f'(x)$ exists.

$$\int_0^t |\,d\phi(u)\,| = O(t)\quad(t\to 0)$$

then

(3.3)
$$\sum_{\nu=1}^{n}\{S\nu(x)-f'(x)\}^2 = o(n\log n)$$

4. In order to prove theorem A we shall require the following lemma.

Lemma. If

(4.1)
$$\int_0^t |\,d\phi(u)\,| = O(t)$$

then

$$\sum_1^n\{S\nu(x)-f'(x)\}^2 = \frac{1}{\pi^2}\int_{1/n}^{\pi}\frac{d\phi(t)}{t^2}\int_{1/n}^{t}\frac{\sin n(u-t)}{u-t}d\phi(u) + O(n).$$

Proof of the Lemma. We have from (3.2)

$$S\nu(x) - f'(x) = \frac{1}{2\pi} \int_0^\pi \frac{\sin(\nu+\frac{1}{2})t}{\sin\frac{1}{2}t} \, d\phi(t)$$

$$= \frac{1}{2\pi} \int_0^{1/n} \frac{\sin(\nu+\frac{1}{2})t}{\sin\frac{1}{2}t} d\phi(t) + \frac{1}{2\pi} \int_{1/n}^\pi \frac{\sin(\nu+\frac{1}{2})t}{\sin\frac{1}{2}t} d\phi(t)$$

$(\nu < n)$.

Now

$$\int_0^{1/n} \frac{\sin(\nu+\frac{1}{2})t}{\sin\frac{1}{2}t} \, d\phi(t) \leq (2\nu+1) \int_0^{1/n} |\,d\phi(t)\,| = O(1)$$

and

$$\int_{1/n}^\pi \cos\nu t \, d\phi(t) \leq \int_{1/n}^\pi |\,d\phi(t)\,| = O(1)$$

Hence

$$S\nu(x) - f'(x) = \frac{1}{2\pi} \int_{1/n}^\pi \sin\nu t \cot\tfrac{1}{2}t \, d\phi(t) + O(1)$$

$$= \frac{1}{\pi} \int_{1/n}^\pi \frac{\sin\nu t}{t} \, d\phi(t) + O(1)$$

and

$$\sum_1^n \{S\nu(x) - f'(x)\}^2 = \frac{1}{\pi^2} \int_{1/n}^\pi \frac{d\phi(t)}{t} \int_{1/n}^\pi \left\{ \sum_1^n (\sin\nu t \, \sin\nu u) \right\} \frac{d\phi(u)}{u} + O(n)$$

$$= \frac{1}{2\pi^2} \int_{1/n}^\pi \frac{d\phi(t)}{t} \int_{1/n}^\pi \sum_1^n \left\{ \cos\nu(u-t) - \cos\nu(u+t) \right\} \frac{d\phi(u)}{u} + O(n)$$

$$= \frac{1}{2\pi^2} \int_{1/n}^\pi \frac{d\phi(t)}{t} \int_{1/n}^\pi \frac{\sin(n+\frac{1}{2})(u-t)}{2\sin\frac{1}{2}(u-t)} \cdot \frac{1}{u} \, d\phi(u)$$

(4.2)

$$- \frac{1}{2\pi^2} \int_{1/n}^\pi \frac{d\phi(t)}{t} \int_{1/n}^\pi \frac{\sin(n+\frac{1}{2})(u+t)}{2\sin\frac{1}{2}(u+t)} \cdot \frac{1}{u} \, d\phi(u) + O(n)$$

Now
$$\int\limits_{1/n}^{\pi}\frac{d\,\phi\,(t)}{t}\int\limits_{1/n}^{\pi}\frac{\cos n\,(u-t)}{u}\,d\,\phi\,(u)\,=\,O\,\{\,(\log n)^2\}\,=\,O\,(n)$$

for, putting $\varphi\,(t)\,=\,\int\limits_{0}^{t}\,|\,d\,\phi\,(u)\,|$ and integrating by parts we get

$$\int\limits_{1/n}^{\pi}\frac{|\,d\,\phi\,(u)\,|}{u}=\left[\frac{\varphi\,(u)}{u}\right]_{1/n}^{\pi}+\int\limits_{1/n}^{\pi}\frac{\varphi\,(u)}{u^2}\,du$$

$$=\,O\,(1)\,+\,O\,(\log n)\,=\,O\,(\log n)$$

Hence from (4.2) we obtain

(4.3)
$$\sum_{1}^{n}\{\,S\nu\,(x)-f'\,(x)\}^2\,=\,\frac{1}{2\pi^2}\int\limits_{1/n}^{\pi}\frac{d\,\phi\,(t)}{t}\int\limits_{1/n}^{\pi}\frac{\sin n\,(u-t)}{u-t}\quad\frac{d\phi\,(u)}{u}$$

$$-\frac{1}{2\pi^2}\int\limits_{1/n}^{\pi}\frac{d\phi\,(t)}{t}\int\limits_{1/n}^{\pi}\frac{\sin n\,(u+t)}{u+t}\quad\frac{d\phi\,(u)}{u}\,+\,O\,(n)$$

$$=\,I_1+I_2+O\,(n)$$

Now, since
$$\frac{1}{u\,(u-t)}=\frac{1}{t}\left\{\frac{1}{u-t}-\frac{1}{u}\right\}$$

and

$$\int\limits_{1/n}^{\pi}\frac{d\phi\,(t)}{t}\int\limits_{t}^{\pi}\frac{\sin n\,(u-t)}{u-t}\quad\frac{d\phi\,(u)}{u}\,=\int\limits_{1/n}^{\pi}\frac{d\phi\,(u)}{u}\int\limits_{1/n}^{u}\frac{\sin n\,(u-t)}{t\,(u-t)}\,d\,\phi\,(t),$$

$$I_1\quad=\frac{1}{2\pi^2}\int\limits_{1/n}^{\pi}\frac{d\phi\,(t)}{t}\int\limits_{1/n}^{t}\frac{\sin n\,(u-t)}{u-t}\quad\frac{d\phi\,(u)}{u}+\frac{1}{2\pi^2}\int\limits_{1/n}^{\pi}\frac{d\phi(t)}{t}\int\limits_{t}^{\pi}\frac{\sin n\,(u-t)}{u-t}\quad\frac{d\,\phi\,(u)}{u}$$

$$=\frac{1}{\pi^2}\int\limits_{1/n}^{\pi}\frac{d\phi\,(t)}{t}\int\limits_{1/n}^{t}\frac{\sin n\,(u-t)}{u-t}\quad\frac{d\phi\,(u)}{u}$$

$$=\frac{1}{\pi^2}\int\limits_{1/n}^{\pi}\frac{d\phi\,(t)}{t^2}\int\limits_{1/n}^{}\left[\frac{1}{u-t}-\frac{1}{u}\right]\sin n\,(u-t)\,d\phi\,(u)$$

(4.4)
$$= \frac{1}{\pi^2} \int_{1/n}^{\pi} \frac{d\phi(t)}{t^2} \int_{1/n}^{t} \frac{\sin n(u-t)}{u-t} \, d\phi(u) + O\left\{ \int_{1/n}^{\pi} \frac{|d\phi(t)|}{t^2} \int_{1/n}^{t} \frac{|d\phi(u)|}{u} \right\}$$

But
$$\int_{1/n}^{\pi} \frac{|d\phi(t)|}{t^2} \int_{1/n}^{t} \frac{|d\phi(u)|}{u} = \int_{1/n}^{\pi} \frac{|d\phi(t)|}{t^2} \left[\left\{ \frac{\varphi(u)}{u} \right\}_{1/n}^{t} + \int_{1/n}^{t} \frac{\varphi(u)}{u^2} \, du \right]$$

$$= \int_{1/n}^{\pi} \frac{|d\phi(t)|}{t^2} \left[\frac{\varphi(t)}{t} + O(1) + O(\log n\, t) \right]$$

$$= O\left\{ \int_{1/n}^{\pi} \frac{\log n\, t}{t^2} |d\phi(t)| \right\}$$

and
$$\int_{1/n}^{\pi} \frac{\log n\, t}{t^2} |d\phi(t)| = \left[\frac{\varphi(t) \log n\, t}{t^2} \right]_{1/n}^{\pi} + 2 \int_{1/n}^{\pi} \frac{\varphi(t) \log n\, t}{t^3} \, dt$$

$$- \int_{1/n}^{\pi} \frac{\varphi(t)}{t^3} \, dt$$

$$= O(n) + O\left\{ \int_{1/n}^{\pi} \frac{\log n t}{t^2} \, dt \right\} + O\left\{ \int_{1/n}^{\pi} \frac{dt}{t^2} \right\}$$

$$= O(n) + O\left\{ n \int_{1}^{n\pi} \frac{\log v}{v^2} \, dv \right\}$$

$$= O(n)$$

Hence from (4.4) we get

(4.5)
$$I_1 = \frac{1}{\pi^2} \int_{1/n}^{\pi} \frac{d\phi(t)}{t^2} \int_{1/n}^{t} \frac{\sin n(u-t)}{u-t} \, d\phi(u) + O(n)$$

Similarly,

(4.6)
$$I_2 = O\left\{ \int_{1/n}^{\pi} \frac{|d\phi(t)|}{t^2} \int_{1/n}^{t} \frac{|d\phi(u)|}{u} \right\} = O(n)$$

Thus from (4.3), (4.5) and (4.6) the lemma follows

5. *Proof of the Theorem.*

$$\sum_{\nu=1}^{n} |S_\nu(x) - f'(x)|^2 = \frac{1}{\pi^2} \int_{1/n}^{\pi} \frac{d\phi(t)}{t^2} \int_{1/n}^{t} \frac{\sin n(u-t)}{u-t} d\phi(u) + O(n)$$

$$= \frac{1}{\pi^2} \int_{1/n}^{\pi} \frac{d\phi(t)}{t^2} \int_{t-m/n}^{t} \frac{\sin n(u-t)}{u-t} d\phi(u)$$

(5.1)
$$+ \frac{1}{\pi^2} \int_{1/n}^{\pi} \frac{d\phi(t)}{t^2} \int_{1/n}^{t-m/n} \frac{\sin n(u-t)}{u-t} d\phi(u) + O(n)$$

$$= J_1 + J_2 + O(n)$$

say.

Now

(5.2)
$$J_2 = O\left\{ \int_{1/n}^{\pi} \frac{|d\phi(t)|}{t^2} \int_{0}^{t} \frac{n}{m} |d\phi(u)| \right\}$$

$$= O\left\{ \frac{n}{m} \int_{1/n}^{\pi} \frac{|d\phi(t)|}{t} \right\} = O\left(\frac{n}{m} \log n \right)$$

because

$$\int_{1/n}^{\pi} \frac{|d\phi(t)|}{t} = \left[\frac{\varphi(t)}{t} \right]_{1/n}^{\pi} + \int_{1/n}^{\pi} \frac{\varphi(t)}{t^2} dt = O(1) + O(\log n).$$

Integrating by parts we get

$$J_1 = \frac{1}{\pi^2} \int_{1/n}^{\pi} \frac{d\phi(t)}{t^2} \left[\left\{ \frac{\phi(u)}{u-t} \sin n(u-t) \right\}_{t-m/n}^{t} - \int_{t-m/n}^{t} \left\{ \phi(u) \frac{d}{du}\left(\frac{\sin n(u-t)}{u-t} \right) \right\} du \right]$$

(5.3)
$$= \frac{1}{\pi^2} \int_{1/n}^{\pi} \frac{n\phi(t)}{t^2} d\phi(t) - \frac{1}{\pi^2} \int_{1/n}^{\pi} \frac{\phi\left(t - \frac{m}{n}\right)}{t^2} \frac{\sin(m)}{m/n} d\phi(t)$$

$$- \frac{1}{\pi^2} \int_{1/n}^{\pi} \frac{d\phi(t)}{t^2} \int_{t-m/n}^{t} \left\{ \phi(u) \frac{d}{du}\left(\frac{\sin n(u-t)}{u-t} \right) \right\} du$$

F. 3

Now, since., as $\phi(t) = o(t)$,

$$\int_{1/n}^{\pi} \frac{\phi(t)}{t^2}\, d\phi(t) = o\left\{ \int_{1/n}^{\pi} \frac{|\, d\phi(t)\,|}{t} \right\}$$

$$= o\left\{ \left[\frac{\varphi(t)}{t} \right]_{1/n}^{\pi} + \int_{1/n}^{\pi} \frac{\varphi(t)}{t^2}\, dt \right\}$$

$$= o\,(\log n),$$

the first two integrals in $(5\cdot3)$ are $o\,(n \log n)$

And

$$\int_{t-m/n} \left\{ \phi(u)\, \frac{d}{du}\left(\frac{\sin n\,(u-t)}{u-t} \right) \right\} du$$

$$= \int_{t-m/n}^{t-1/n} \phi(u) \left\{ \frac{n \cos n\,(u-t)}{u-t} - \frac{\sin n\,(u-t)}{(u-t)^2} \right\} du$$

$$+ \int_{t-1/n}^{t} \phi(u) \left\{ \frac{n \cos n\,(u-t)}{u-t} - \frac{\sin n\,(u-t)}{(u-t)^2} \right\} du$$

$$= \int_{t-m/n}^{t-1/n} o\,(u)\, \frac{n}{u-t}\, du + \int_{t-1/n}^{t} o\,(u)\, n^2\, du$$

$$= o\,(nt)$$

so that the last integral in $(5\cdot3)$ is $o\,(n \log n)$

Hence

(5.4) $J_1 = o\,(n \log n)$

when m is fixed. If we take m sufficiently large (5.2) will mean that $J_2 = o\,(n \log n)$

The required result now follows from (5.1), (5.2) and (5.4).

REFERENCES

1. Marcinkiewicz, J. 1939, *Jour. Lond. Math. Soc.* 14, 162-168.
2. Young, W. H. 1913, *Proc. Lond. Math. Soc.* XIII, 21-23.
3. Zygmund, A. *Trigonometrical Series* (Warsaw 1935).

ON THE COMPOSITIONS OF THE NEGATIVELY AND THE POSITIVELY CHARGED COLLOIDAL SOLUTIONS OF VARIOUS FERRIC SALTS AND THEIR DEPENDENCE ON THE DISSOCIATION CONSTANTS OF THEIR POLYBASIC ACIDS

By S. P. Mushran and Satya Prakash

Chemical Laboratories, University of Allahabad

(*Read on 14th September* 1946.)

Grimaux[1], Holmes and coworkers[2], prepared and studied the positively charged sols of ferric arsenate, ferric phosphate and chromic arsenate. Varma and Prakash[3] studied the peptisation of ferric arsenate, phosphate, molybdate and tungstate by ferric chloride in presence of various peptising agents which gave positively charged sols. Ferric borate sol and gel were obtained by Prakash and Dhar[4]. Much less work, however, appears to have been done on the sols of these ferric salts, bearing a negative charge. In several publications[5], we have described the preparation of the negatively charged colloidal solutions of various ferric salts and this paper records our observations on the compositions of the negatively charged colloidal solutions of ferric vanadate, molybdate, tungstate, borate, arsenate and phosphate. For comparison, we have also studied the compositions of the positively charged sols of the corresponding ferric salts. An attempt has also been made to discuss the compositions in the light of the dissociation constants of the various polybasic acids.

EXPERIMENTAL

Preparation of the positively charged sols :—A large number of sols (two of each denoted by A and B were prepared by adding to ferric chloride solution different amounts of precipitants such as ammonium vanadate, potassium molybdate, sodium tungstate, sodium borate, potassium dihydrogen arsenate and potassium dihydrogen phosphate and dissolving the precipitates in excess ferric chloride. The sols were purified by dialysis.

Preparation of the negatively charged sols :—A large number of sols (two of each denoted by A and B) were prepared by adding to ferric chloride

solution different amounts of the same precipitants as above and dissolving the precipitates in caustic soda in presence of glucose or glycerine. The sols were purified by dialysis.

COMPOSITION OF THE SOLS

Ferric vanadate sol :—The amount of iron in the sol was estimated by reducing ferric iron to the ferrous state by the addition of a freshly prepared stannous chloride solution and titrating against standard potassium dichromate solution. In order to find out how much of the vanadate is in the combined state with iron, the sol was coagulated cataphoretically as well as by potassium chloride. The coagulum was in both the cases separately collected, thoroughly washed and estimated for vanadate by dissolving it in hydrochloric acid and reducing the pentavalent vanadium to the quadrivalent state by means of ferrous ammonium sulphate and titrating the excess ferrous ammonium sulphate against potassium dichromate using potassium ferricyanide as an external indicator. The combined iron corresponding to this amount of vanadate was calculated on the assumption that the ferric vanadate is $Fe(VO_3)_3$ in the case of the negatively charged sol and $FeVO_4$ in the case of the positively charged sol. The rest of the iron is present as hydrated ferric oxide. From the ratio of the free to the combined iron, the empirical formula of the sol was calculated.

Ferric molybdate sol :—The amount of iron in the sol was estimated by the use of the reagent cupferron ($C_6H_5N.NO.ONH_4$). The precipitation was carried out in cold acidic solution. The precipitate was washed with cold water and then with 6N—ammonia solution to convert the iron precipitate into ferric hydroxide, which was ignited and estimated as Fe_2O_3. In order to find out how much of the molybdate is in the combined state with iron, the sol was coagulated cataphoretically as well as by potassium chloride. The coagulum in both the cases was separately collected, thoroughly washed and estimated for molybdate as MoO_3. This was done by dissolving the coagulum in hydrochloric acid and precipitating molybdenum as MoS_3 in a small pressure flask by hydrogen sulphide. The flask was then corked, heated over a water bath until the precipitate completely settled. The precipitate was filtered off when it became cold and was washed with dilute acid solution. The moist filter paper was placed in a silica crucible, dried upon a water bath and by gentle ignition

was changed into oxide and weighed as such. The combined iron corresponding to this amount of molybdate was calculated on the assumption that the ferric molybdate is $Fe_2(MoO_4)_3$. The rest of the iron is present as hydrated ferric oxide. From the ratio of the combined to the free iron present, the empirical formula of the sol was calculated.

Ferric tungstate sol :—The amount of tungsten in the combined state with iron was found by coagulating the sol cataphoretically and also by potassium chloride ; the coagulum was in both the cases separately collected, washed and estimated for tungsten by precipitating tungsten as tungstic acid by means of concentrated hydrochloric acid and concentrated nitric acid. The tungstic acid was allowed to settle down and then filtered. The precipitate was washed with a solution of ammonium chloride to prevent the formation of colloidal solutions. The precipitate was slowly ignited in a platinum crucible and the residue was weighed as WO_3. After precipitating tungstic acid and filtration, to the filtrate was added strong ammonia solution and iron was precipitated as ferric hydroxide. This was ignited in a platinum crucible and estimated as Fe_2O_3. The combined iron corresponding to the amount of combined tungstate was calculated on the assumption that the ferric tungstate is $Fe_2(WO_4)_3$. The rest of the iron is present as hydrated ferric oxide. From the ratio of the free to the combined iron, the empirical formula of the sol was calculated.

Ferric borate sol :—The amount of iron was estimated by the use of the reagent cupferron (cf. molybdate sol). In order to find out how much of the borate was in the combined state with iron, the sol was coagulated with potassium chloride as well as cataphoretically. The total coagulum in each case was separately collected, washed and estimated for borate. This was done by dissolving the coagulum in hydrochloric acid and the solution was made colourless by reducing the ferric iron to the ferrous state by adding just sufficient amount of stannous chloride. This solution was made distinctly alkaline by the addition of a caustic soda solution using phenolphthalein as an indicator. It was then made faintly acid with 0·05 N. HCl, and its colour was brought to faint pink with 0·05 N. NaOH. One third of its volume of glycerine was now added to the liquid and it was again titrated with 0·05N NaOH solution until it became pink. 10 c.c. more of glycerine were added which caused the solution to become colour-

less. Again sodium hydroxide was added and the process was repeated until finally glycerine caused no further action on the end point. Each millilitre of the 0.05N. NaOH used in this last titration indicates the presence of 0.0031g. of H_3BO_3 in the volume of sol taken. The combined iron corresponding to this amount of borate was calculated on the assumption that the ferric borate is $FeBO_3$. The rest of the iron is present as hydrated ferric hydroxide. From the ratio of the free to the combined iron, the empirical formula of the sol was calculated.

Ferric arsenate sol :—The amount of iron was estimated by the use of the reagent cupferron (cf. molybdate sol). A known volume of the sol was coagulated by potassium chloride as well as cataphoretically. The coagulum was in both the cases separately collected, washed and the arsenate was estimated as arsenic pentasulphide. For this, the coagulum was collected in an Erlenmeyer's flask which was cooled by surrounding it with ice. The coagulum was dissolved by adding to it a well cooled solution of concentrated hydrochloric acid little by little. A very rapid stream of hydrogen sulphide was then conducted in this solution until it was saturated with the gas. The flask was stoppered and was allowed to stand for two hours. The arsenic pentasulphide was filtered off through a Gooch crucible and washed thoroughly with water. After drying at $105°$C, the arsenic was weighed as As_2S_5. The combined iron corresponding to this amount of arsenate was calculated on the assumption that the ferric arsenate is $FeAsO_4$. The rest of the iron is present as hydrated ferric oxide. From the ratio of the free to the combined iron, the empirical formula of the sol was calculated.

Ferric phosphate sol :—The amount of iron was estimated by dissolving the sol in hydrochloric acid and reducing ferric iron to the ferrous state by the addition of freshly prepared stannous chloride solution and titrating against standard potassium dichromate solution. In order to find out how much of the phosphate is in the combined state with iron, the sol was coagulated cataphoretically as well as by potassium chloride. The coagulum in each case was separately collected, washed and estimated for phosphate as phosphomolybdate. For this, the coagulum was dissolved in concentrated nitric acid and this solution was heated nearly to boiling. A three per cent solution of ammonium molybdate was run into the phosphate solution

slowly in a fine stream. The precipitate of phosphomolybdate was allowed to stand for nearly an hour in a warm place and then filtered through a Gooch crucible. It was washed with one per cent. solution of nitric acid and finally with water. The precipitate was dried in an oven and weighed as phosphomolybdate. The combined iron corresponding to this amount of phosphate was calculated on the assumption that the ferric phosphate is $FePO_4$. The rest of the iron is present as hydrated ferric oxide. From the ratio of the free to the combined iron, the empirical formula of the sol was calculated.

The viscosities of the sols were measured by the Ostwald's viscometer at 30 °C. Knowing the viscosity of water at 30 °C, the amount of water bound per litre of the sols was calculated from the Hatschek's equation [6] expressed in the following form :

$$\text{Water bound} = \frac{1000}{A} = 1000 \left(\frac{\eta_s - \eta_w}{\eta_w} \right)^3$$

where A is the ratio of the total volume of water in the sol to the volume of water bound, η_s is the viscosity of the sol at 30 °C and η_w is the viscosity of water at the same temperature.

In the following tables are recorded the results of analysis and the compositions of the various sols studied in this paper.

TABLE I

Positively charged sols

Per litre :—

Sol		Total iron g.	Combined acid g.	*Combined iron g.	Free iron g	η_s x 10^{-3}	Water Bound g.	Empirical formula
Vanadate	(A)	7.3708	2.7285	1.6750	5.6958	8.58	0.2634	$17Fe_2O_3.10FeVO_4.5H_2O$
	(B)	5.5282	3.1832	1.9540	3.5742	8.64	0.3519	$9Fe_2O_3.10FeVO_4.6H_2O.$
Molybdate		16.4638	9.7000	2.5080	13.9558	8.76	0.5786	$11Fe_2O_3.2Fe_2(MoO_4)_3$ $3.H_2O$
Tungstate	(A)	7.8225	8.3000	1.3310	6.4915	8.55	0.2249	$5Fe_2O_3.Fe_2(WO_4)_3.H_2O$
	(B)	7.5815	16.1700	2.5930	4.9885	8.60	0.2913	$2Fe_2O_3.Fe_2(WO_4)_3.H_2O$
Borate	(A)	8.7495	1.9220	1.7360	7.0135	8.40	0.0854	$2Fe_2O_3.FeBO_3.0.2H_2O$
	(B)	9.0220	1.7360	1.5680	7.4540	8.38	0.0729	$5Fe_2O_3.2FeBO_3.0.3H_2O$
Arsenate	(A)	11.8478	7.1320	5.3128	6.5350	9.00	1.2520	$3Fe_2O_3.5FeAsO_4.4H_2O$
	(B)	17.4151	13.8582	10.3234	7.0917	9.28	2.4400	$Fe_2O_3.3FeAsO_4.2H_2O$
Phosphate	(A)	15.0768	19.4000	11.4000	3.6768	8.72	0.4953	$Fe_2O_3.6FePO_4.H_2O$
	(B)	13.7366	15.2500	8.9600	4.7760	8.62	0.3207	$5Fe_2O_3.19FePO_4.2H_2O$

TABLE II

Negatively charged (glucose) sols.

Per litre :—

Sol		Total iron g.	Combined acid g.	*Combined iron g.	Free iron g.	η_s x 10^{-3}	Water Bound g.	Empirical formula
Vanadate	(A)	1·3402	0·4093	0·0837	1·2565	8·28	0·0219	$15Fe_2O_3.2Fe(VO_3)_3.2H_2O$
	(B)	1·3122	0·4547	0·0930	1·2192	8·30	0·0344	$13Fe_2O_3.2Fe(VO_3)_3.2H_2O$
Molybdate	(A)	3·1823	0·5800	0·1499	3·0324	8·32	0·0424	$20Fe_2O_3.Fe_2(MoO_4)_3.2H_2O$
	(B)	2·7816	0·6000	0·1551	2·6264	8·35	0·0563	$17Fe_2O_3.Fe_2(MoO_4)_3.2H_2O$
Tungstate	(A)	1·5177	0·8400	0·1348	1·3829	8·20	0·0891	$10Fe_2O_3.Fe_2(WO_4)_3.$ $0.4H_2O$
	(B)	2 0562	1·3100	0 2102	1·8460	8·28	0·2186	$9Fe_2O_3.Fe_2(WO_4)_3.$ $0.7H_2O$
Borate	(A)	1·3083	0·1240	0·1120	1·1963	8·34	0·0514	$5Fe_2O_3.FeBO_3.H_2O$
	(B)	1·2952	0·1240	0·1120	1·1832	8·35	0·0563	$11Fe_2O_3.2FeBO_3.H_2O$
Arsenate	(A)	1·1050	0·2126	0·1584	0·9466	8·30	0·0344	$3Fe_2O_3.FeAsO_4.H_2O$
	(B)	1·5246	0·2416	0·1799	1·3477	8·32	0·0424	$4Fe_2O_3.FeAsO_4.H_2O$
Phosphate	(A)	1·1052	0·3264	0·1918	0·9134	8·25	0·0190	$5Fe_2O_3.2FePO_4.H_2O$
	(B)	0·8097	0·5442	0·3198	0·4899	8·27	0·0244	$3Fe_2O_3.4FePO_4.H_2O$

TABLE III

Negatively charged (Glycerine) sols.

Per litre :

Sol		Total iron g.	Combined acid g.	*Combined iron g.	Free iron g.	η_s x 10^{-3}	water bound g.	Empirical formula
Vanadate	(A)	1·2843	0·4729	0·0967	1·1876	8·35	0·0563	$6Fe_2O_3.Fe(VO_3)_3.H_2O$
	(B)	2·5686	0·8167	0·1669	2·4017	8·39	0·0790	$7Fe_2O_3.Fe(VO_3)_3.H_2O$
Molybdate	(A)	3·2127	0·5200	0·1344	3·0783	8·36	0·0615	$23Fe_2O_3.Fe_2(MoO_4)_3$ $.3H_2O$
	(B)	2·9976	0·4000	0·1034	2·8942	8·72	0·4953	$28Fe_2O_3.Fe_2(MoO_4)_3$ $.30H_2O$
Tungstate	(A)	1·5624	0·7119	0·1142	1·4482	8·42	0·0994	$13Fe_2O_3.Fe_2(WO_4)_3.$ $1.5H_2O$
	(B)	2·8883	1·0000	0·1604	2·7279	8·66	0·3850	$17Fe_2O_3.Fe_2(WO_4)_3$ $.15H_2O$
Borate	(A)	1·3125	0·1550	0·1319	1·1806	8·28	0·0219	$9Fe_2O_3.2FeBO_3.H_2O$
	(B)	1·3864	0·1421	0·1284	1·2580	8·92	0·9931	$5Fe_2O_3.FeBO_3.24H_2O$
Arsenate	(A)	1·4128	0·2420	0·1803	1·2325	8·30	0·0344	$7Fe_2O_3.2FeAsO_4.H_2O$
	(B)	1·4331	0·2607	0·1942	1·2389	8·34	0·0514	$3Fe_2O_3.FeAsO_4.H_2O$
Phosphate	(A)	1·1168	0·5584	0·3281	0·7887	8·40	0·0854	$12Fe_2O_3.10FePO_4.8H_2O$
	(B)	0·8376	0·5949	0·3497	0·4879	8·44	0·4470	$7Fe_2O_3.10FePO_4.10H_2O$

*Combined acid in vanadate sols corresponds to V_2O_5, in molybdate sols to MoO_3, in tungstate sols to WO_3, in borate sols to H_3BO_3, in arsenate sols to As, and in phosphate sols to PO_4.

DISCUSSION

The composition of the sols depend upon a large number of factors, viz., (i) the initial concentrations of the reactants, (ii) the amount cf peptising agents used, (iii) the amount of alkali used, (iv) the extent of dialysis and (v) the extent to which the precipitated ferric salts are hydrolysed.

Besides these factors, the compositions would also differ with respect to the adsorption capacity of various acid oxides by ferric oxide.

Taken from the existing data in literature the dissociation constants of phosphoric[7], arsenic[8], vanadic[9], tungstic[9], molybdic[9] and boric[10] acids are:

Acid	Dissociation constant
Phosphoric	$1 \cdot 1 \times 10^{-2}$
Arsenic	$5 \cdot 0 \times 10^{-3}$
Vanadic	$4 \cdot 0 \times 10^{-3}$
Tungstic	$7 \cdot 6 \times 10^{-5}$
Molybdic	$5 \cdot 7 \times 10^{-7}$
Boric	$6 \cdot 6 \times 10^{-10}$

The dissociation constants of these acids are in the following order:
Phosphoric > Arsenic > Vanadic > Tungstic > Molybdic > Boric.... (1)

The stability of the ferric salts would very much depend on the dissociation constants of the acids. In the following table, we are recording the proportion of ferric oxide to ferric salts of the various positively and negatively charged colloidal solutions studied in this paper. It is natural that a salt more stable will be less hydrolysed and therefore, in its composition there would be a less proportion of Fe_2O_3 to the ferric salt of the polybasic acid.

TABLE V

Sol		Positive	Glucose	Negative Glycerine	
Vanadate	(A)	3·4 : 1	15 : 1	14 : 1	
	(B)	1·8 : 1	13 : 1	12 : 1	
Molybdate	(A)	5·5 : 1	17 : 1	28 : 1	
	(B)	—	20 : 1	23 : 1	
Tungstate	(A)	5·0 : 1	10 : 1	17 : 1	
	(B)	2·0 : 1	9 : 1	13 : 1	
Borate	(A)	4·0 : 1	11 : 1	10 : 1	
	(B)	5·0 : 1	10 : 1	9 : 1	
Arsenate	(A)	1·2 : 1	6 : 1	7 : 1	
	(B)	6·6 : 1	8 : 1	6 : 1	
Phosphate	(A)	0·3 : 1	5 : 1	2·4 : 1	
	(B)	0·5 : 1	1·5 : 1	1·4 : 1	

F. 4

From the composition of the sols, it is clear that the polybasic acids are in the combined form with iron in the above sols in the following order :

Positively charged sols—

Phosphoric > Arsenic > Vanadic > Tungstic > Molybdic – Boric . . . . (2)

Negatively charged sols—

Phosphoric>Arsenic>Boric>Vanadic— Tungstic>Molybdic . . . (2)

Comparing series (1) and (2) it is interesting to note that the stability of the sols follow about the same order as the dissociation constants except for the borate sol where there seems to be a departure. Boric acid seems to be greatly adsorbed by the active surface of ferric hydroxide.

A perusal of table V shows that the positively charged sols contain proportionately a higher content of iron in combined state with the corresponding polybasic acids. The negatively charged sols were prepared mainly in the alkaline medium. Even after dialysis, the pH of the supernatant liquids obtained after the coagulation of the sols with electrolytes lie between 7·03 and 7·64. The negatively charged sols prepared were sufficiently alkaline. This accounts for the larger proportion of ferric oxide in the negatively charged sols.

SUMMARY

1. The compositions of the positively and the negatively charged colloidal solutions of ferric vanadate, molybdate, tungstate, borate, arsenate and phosphate have been studied.

2. It is observed that the positively charged sols contain proportionately a higher content of iron in combind state with the corresponding polybasic acid than the negatively charged sols, where there is a larger content of ferric oxide.

3. It has been shown that the stability of the sols follow about the same order as the dissociation constants of the polybasic acids, except for the borate sols where there seems to be an anomaly. It appears that boric acid is greately adsorbed by the active surface of ferric hydroxide.

REFERENCES

1. Grimaux: Compt. Rend., **98**, 1540 (1884).
2. Holmes and coworkers: J. Amer. Chem. Soc., **38**, 1970, (1916); **40**, 1014 (1918); **41**, 763, (1919).
3. Varma and Prakash: Z. anorg. Chem., **205**, 241 (1932).
4. Prakash and Dhar: J. Ind. Chem. Soc., **7**, 367 (1930).
5. Mushran and Prakash: J. Phys. Chem. **50**, 251 (1946). J. Ind. Chem. Soc., **23**, 111. (1946). Mushran: Current Science, **14**, 123,200,233 (1945); **15**, 24 (1946).
6. Hatschek: Kolloid-Z., **8**, 34 (1911).
7. Abbot and Bray: J. Amer. Chem. Soc., **31**, 729 (1909).
8. Blanc: J. Chim. Phys., **18**, 28 (1920).
9. Bhagwat and Dhar: J. Indian Chem. Soc., **6**, 807, (1929).
10. Lunden: J. Chim. Phys, **5**, 574, (1907).

PREPARATION OF SILVER SOL BY THE REDUCTION OF ORGANIC SALTS OF SILVER. PART I

By B. Chakravarti and S. Ghosh

Department of Chemistry, University of Allahabad

Kohlschütter[1] prepared colloidal silver by passing hydrogen through a saturated solution of silver oxide at temperatures between 50°. and 60°. He reported that the nature of the material of the containing vessel markedly affects the colour of the silver sol formed. H. B. Weiser and M. F. Roy[2] and others using ultra filtered solution of silver oxide and pure hydrogen found that no sol can be formed by this method and they attributed this to the absence of the nucleii of silver which is the main factor responsible for the formation of silver sol in Kohlschütter's experiment.

In the present investigation we have attempted to prepare silver sols by the reduction of organic silver salt solutions by different reducing gases. A few of the preliminary experiments showed that hydrogen gas was more efficient for the preparation of the sol. In the following table our observations in the reductions of different silver salt solutions by a current of hydrogen both in cold and hot are recorded :—

TABLE I

Silver salt Solutions	Observations
Formate	No sol formed but a black precipitate, the reduction being immediate.
Acetate.	Same as above, the reduction being slower
Oxalate.	Do.
Tartrate.	A beautiful red sol formed.
Citrate.	Do.

EXPERIMENTAL

Silver citrate obtained by the interaction of silver nitrate and sodium citrate was washed free from electrolytes. The precipitate was suspended in conductivity water in a pyrex flask in water baths maintained at different

temperatures, and pure hydrogen was passed and a sol of silver was formed.

For the estimation of the concentration of the sol the liquid containing 10·0 c. c. the sol was taken in two separate centrifugal tubes and kept in baths maintained at 25 °C. for fifteen minutes and then centrifuged at 2500 revolutions per minute for five minutes in order to remove any suspended material. 5 c.c. of the supernatant liquid from one of the tubes was estimated for the silver content whilst the other was treated with a crystal of potassium nitrate to precipitate the sol. The precipitate was removed by centrifuging and 5 c.c. of the clear supernatant liquid was estimated for the silver content by titrating against a standard thiocynate solution for estimating the amount of silver in the dissolved state. The following table gives the comparative silver content of sols fromed by the reduction of silver citrate solutions at 50°. and 70°. :—

TABLE II

Properties.	Sol A.	Sol B.
Temperature.	50°.	70°.
Hours gas passed.	21	11
Total Silver (mgs per-lit)	302·05	1164·00
Colloidal	112·20	1095·96

The above results show that the reduction is faster and more complete if the temperature of the reduction is high. .

With the difference in the rate of reduction the colours of the sols formed were different. For the sols prepared at lower temperature the colour was greenish blue whilst for the sol prepared at higher temperature the colour was red. The colours developed by the sols were determined by measuring the extinction coefficients at different wavelengths of the visible spectrum with a Nuttings Sprectro-photometer, the actual extinction coefficient D/t being derived by using Bunsen Roscoe's formula $D/t = \log_{10} \tan \theta$, where θ is the angular reading and 't' the thickness of the absorbing medium and 'D' the reading on the density scale. The cell used had a thickness of 1 c.m. Hence the reading on the density scale directly gives the extinction coefficient.

The sol was found to follow Beer's Law.

TABLE III

The sol under consideration being Sol A. and different volumes of it diluted to 10. c.c. and extinction coefficient measured.

Wavelength of light in A.°U.	Volume of sol made to 10 c. c. by dilution.		
	2·0 c.c.	1·0 c.c.	0·5 c.c.
5200	1·30	0·66	0·32
5400	1·28	0·64	0·32
5600	1·20	0·60	0·30
5800	1·20	0·60	0·30
6000	1·20	0·60	0·30
6200	1·18	0·60	0·28
6400	1·18	0·58	0·26

In the following table the extinction coefficients of the two sols A and and B containing the same amount of silver in colloidal form are given.

TABLE IV

Wavelength of light in A.°U.	Sol A	Sol B
5200	5·1	3·3
5400	5·0	2·5
5600	4·6	2·0
5800	4·6	1·6
6000	4·6	1·4
6200	4·6	1·3
6400	4·2	1·1

The figures for the extinction coefficient show that the sol prepared at lower temperature is of greenish colour as evinced by the fact that the extinction coefficient is nearly constant after 5600 A.°U. while the sol prepared at higher temperature is definitely red as shown by the decrease in the extinction coefficient from 5200 A.°U. to 6400 A°.U.

In order to study the rate of formation of sols of silver and the chemical kinetics of reduction, solutions of silver citrate were reduced by passing a current of hydrogen gas and the reduction was carried to the same extent to have an idea of the difference in the nature of the sol with the rise in the temperature of reduction. The reduction and the estimations were carried on as before.

Our results show that the reduction of silver citrate solution by passing a current of hydrogen is unimolecular with an induction period, which is especially developed for the reduction at lower temperature. The temperature coefficient of the reduction is approximately 2.

It has been found that these sols also obey Beer's Law on dilution for the absorption in the visible part of the spectrum and in the following table the calculated amount of the extinction coefficient for the sols of the same silver content is calculated.

TABLE V

Wavelength of light in A°U.	Silver citrate reduced at 50°C	70°C
5200	1·0	1·1
5400	1·0	1·0
5600	1·1	0·9
5800	1·0	0·8
6000	1·0	0·7
6200	1·0	0·6
6400	0·9	0·5

We conclude from our results that with the rise in the temperature of reduction the colour of the sol tends to be red i.e., the size of the colloidal particle is smaller at higher temperatures than at lower temperatures Scaum and Lang[3] prepared silver sol by reducing silver nitrate with sodium sulphide and phenylene diamine and determined the influence of size of particle on the colour of the sol. They reported that the colour of silver hydrosol will be orange, red purple, violet, blue violet, blue and blue green with an increase in the size of the particles.

Our experimental results on the formation of metal silver by the reduction of silver citrate solution show that greater the speed of reduction the smaller is the size of the colloidal particles formed. This, however, is not always true for we have observed that the precipitation of silver from its organic salts such as formate and acetate by passing a current of hydrogen is fairly rapid, but no sols are formed. On the other hand, sols are easily obtained by the reduction of organic salts of polyvalent anions as tartrate and citrate. We are of the opinion that besides the speed of reduction controlling the size of the particles precipitated, the presence of ions which can be absorbed on the surface to impart the charge to the colloid particle is important.

REFERENCES

1.   Kohlschutter. 1901, Z. Electro Chem., **49**, 14.
2.   Weiser and Roy. 1933, J. Physical Chemistry, **37**.
3.   Schaum and Lang. 1921, Koll Z. **28**, 243.

QUANTITATIVE STUDY OF THE EXTENT OF INHIBITION OF THE PRECIPITATION OF STANNIC SULPHIDE BY OXALIC ACID AND OXALATES

By Arun K. Dey

DEPARTMENT OF CHEMISTRY, UNIVERSITY OF ALLAHABAD

AND

A. K. Bhattacharya,

DEPARTMENT OF CHEMISTRY, UNIVERSITY OF SAVGOR

ABSTRACT

The inhibition of the precipitation of stannic sulphide from acidic solutions of stannic chloride, in presence of different amounts of oxalic acid and potassium oxalate, has been studied quantitatively. It has been found that soluble oxalates are far better inhibitors than the acid. The inhibition has been ascribed to be due to the complex fromation between stannic tin and oxalate radical.

The inhibition of the precipitation of stannic sulphide in presence of oxalic acid was reported by Clarke[1], in 1903, and Rössing[2] devolved a method for the quantitative separation of tin and antimony, from a mixture containing these metals, by the use of oxalic and tartaric acids. Not much quantitative observation has been recorded on this phenomenon. In this communcation, we are recording the results on the quantitative study of the inhibition in presence of various quantites of oxalic acid and potassium oxalate.

EXPERIMENTAL

A standard solution of stannic chloride was prepared by dissolving Schuchardt's sample of stannic chloride crystals in normal hydrochloric acid solution, and the amount of tin in the solution was estimated as stannic oxide. Standard solutions of oxalic acid and potassium oxalate were prepared by dissolving the required quantities of pure oxalic acid and potassium oxalate in water. A solution of sodium sulphide of $M/5$ strength was prepared by estimating iodometrically. It was noted that

86

M/5 was the minimum strength of sodium sulphide that could precipitate stannic sulphide completely, from the prepared solution of stannic chloride. Since the stannic chloride solution contained normal hydrochloric acid, the resultant mixture after the addition of an equal volume of sodium sulphide, remained acidic.

Several test tubes were taken, and in each 10 c.c. of the stannic chloride solution (strength M/12·34) were delivered. To each were added 10 c.c. of oxalic acid or the oxalate solution of known concentrations, and then 10 c.c. of the sodium sulphide solution. The first was filtered immediately, the second after half an hour, the third after four hours, the fourth after eight hours, and the fifth after ninety-six hours. Different sets of observations were taken, using different concentrations of the oxalic acid or the oxalate solutions. A blank experiment using water in presence of the oxalic acid or the oxalate was also performed.

The precipitated stannic sulphide was estimated by filtering as such, and then weighing as oxide, by the ignition of the sulphide over a blast lamp. The filtrate from each of the experiments were boiled, and 1 c.c. of the sodium sulphide solution added. This was left overnight, and the precipitated stannic sulphide was filtered and estimated as stannic oxide. The sum of the amounts of stannic oxide estimated in the residue and the filtrate corresponded with that available from the total amount of stannic chloride taken.

In tables I and II the results obtained on the inhibition of the precipitation by different amounts of oxalic acid and potassium oxalate are recorded.

TABLE I

Percentage inhibition in presence of different concentrations of oxalic acid solutions.

Ratio $Sn^{IV} : C_2O_4''$	Time after which filtration was done				
	0 hour	½ hour	4 hours	8 hours	96 hours
1 : 0	2·35%	2·35%	1·90%	—	—
1 : 0·778	3·08	2·35	2·35	0·29%	0·29%
1 : 1·542	3·66	3·08	2·78	0·81	0.66
1 : 3·085	6·30	4·98	4·18	1·50	1·32
1 : 3·526	73·40	7·18	6·23	2·93	1·54
1 : 4·113	73·50	7·33	6·30	3·00	1·54
1 : 6·170	73·50	7·33	6·45	3·15	1·54

F. 5

In all the cases the inhibition is at first more marked, but the precipitate soon appears. After a period of 96 hours, most of the tin gets precipitated as stannic sulphide.

TABLE II

Percentage inhibition in presence of different concentrations of potassium oxalate solutions.

Ratio $Sn^{IV} : C_2O_4''$	Time after which filtration was done				
	0 hour	$\frac{1}{2}$ hour	4 hours	8 hours	96 hours
1 : 0	2·35%	2·35%	1·90%	—	—
1 : 0·389	9·74	9·29	9·22	8·78%	8·72%
1 : 0·778	23·22	19·93	19·05	18·54	18·31
1 : 1·542	34·11	29·28	26·08	19·49	19·32
1 : 2·075	50·15	32·06	28·18	21·61	20·58
1 : 2·468	63·37	39·38	32·21	22·91	21·96
1 : 3·085	87·63	40·56	35·21	23·35	22·04

For comparison, we have recorded below the percentage inhibition by oxalic acid and potassium oxalate for the zero hour readings.

TABLE III

Comparison of the percentage inhibition by oxalic acid and potassium oxalate.

Ratio	Oxalic acid	Potassium oxalate
1 : 0	2·35 %	2·35 %
1 : 0·778	3·08	23·22
1 : 1·542	3·66	34·11
1 : 3·085	6·30	87·63

These results show the remarkable efficiency of potassium oxalate as an inhibitor in the precipitation of stannic sulphide. Though the inhibiting power of oxalic acid was known since long, nothing was on record regarding the same property of oxalates. We[3] have been the first to report this fact, and have further shown the remarkable efficiency of this inhibitor.

With increasing concentrations of oxalic acid and potassium oxalate the precipitate showed a wide range of colour, beginning from yellow to deep brown, where it becomes gelatinous. Here the precipitate closely resembles a freshly obtained precipitate of ferric hydroxide in appearance.

As the quantity of oxalate ions in the solution are increased, further, the system assumes a colloidal character, and passes through the filter paper almost completely.

In another set of experiments, the stability of the complex was studied by allowing stannic chloride and oxalate solutions to remain in contact for different lengths of time, and then stannic sulphide was precipitated by the addition of sodium sulphide. The precipitates obtained were filtered immediately after the addition of sodium sulphide, and stannic oxide estimated in the precipitate and the filtrate as before. From the experiments it can be inferred that the inhibition does not depend on the period for which the mixture of solutions of stannic chloride and oxalic acid or potassium oxalate are allowed to stand. Thus the complex formed in these cases is quite stable, and has no tendency to decompose even when kept for ninety-six hours.

The gradual settling down of the sulphide precipitate with time, as seen from tables I and II, can be ascribed to the formation of a colloidal solution of stannic sulphide, which slowly coagulates. It is well known, that the formation of a colloid often precedes a chemical reaction, prior to the complex formation. Until a definite excess of the inhibitor is present, the complex formation is not complete, and hence inhibition is only partial. It has already been reported that small amounts of the inhibitors are not able to bring about the inhibition of the precipitation of stannic sulphide, but appreciable quantities are required.

The first column in tables I and II, denotes the zero hour action of sodium sulphide on the complex (plus the stability of the complex). The fifth column, of course, conveys an idea of the amounts of complex formed, which is not decomposed even by the prolonged action of sodium sulphide.

The following explanations are advanced regarding the gradual settling of the precipitated stannic sulphide in presence of oxalic acid or potassium oxalate :

(i) The solvent action of hydrochloric acid present in the system, opposes the precipitating action of sodium sulphide, but we observed that the solvent action of the acid becomes negligible where the action of sodium sulphide is prolonged for eight hours, when the whole amount of tin is precipitated as sulphide.

(ii) Apparently it might appear that sodium sulphide dissolves stannic sulphide, but since the mixture remains finally acidic (due to the hydrochloric acid present), this possibility is ruled out.

(iii) Coagulation proceeds with time, and it takes eight hours for the precipitated stannic sulphide to settle completely. In cases where oxalate ions are present, the time of precipitate formation is prolonged, and it takes as much as ninety-six hours to settle. The rest of tin is, however. present as complex, and is never precipitated, until and unless boiled, this being more prominent with potassium oxalate.

We conclude that the inhibition of the precipitation of stannic sulphide, from soluble stannic salt solutions, by oxalic acid or soluble oxalates arises from the formation of complex compounds, this being more prominent with an oxalate than oxalic acid. It should be mentioned here that several earlier workers[4-7] have isolated complex salts between stannic salts and the oxalates. We have adopted some physico-chemical methods for the study of the composition and formulae of the complexes formed and the results obtained therein are in agreement with the view that potassium oxalate forms complexes more easily with stannic chloride than is done by oxalic acid.

REFERENCES

1. Clarke—Chem. News, 1903, 21, 124.
2. Rössing—Zeit. anal. Chem., 1905, 13, 1.
3. Dey and Bhattacharya—Current Science, 1945, 14, 70.
   Dey—University of Allahabad Studies, 1946, 12, 6.
4. Rosenheim and coworker—Zeit. anorg Chem., 1899, 20, 281.
5. Hausmann and Löwenthal—Leib. Ann., 1854, 89, 104.
6. Pechard—Compt. Rend., 1893, 116, 1513.
7. Rammelsberg—Pogg. Ann., 95, 117.; J. B. 1855, 465.

MOTION OF A COMPRESSIBLE FLUID WITH VARYING VISCOSITY GIVEN BY $\mu=\mu_0 + \epsilon_1\, x$ FOR POSITIVE VALUES OF x, WITH REFERENCE TO SOUND WAVES.

By Santi Ram Mukherjee

Engineering College, Benares Hindu University

Communicated by Dr. P. L. Shrivastava, M. A. D. Phil., (Oxon).

Department of Mathematics, Allahabad University

In the investigations which follow, the thermal processes are neglected for simplicity. It is hoped that the thermal process will be taken into consideration in a later paper. In this paper we shall investigate the motion of fluid in which the law of variation in coefficient of viscosity is given by $\mu=\mu_0 + \epsilon_1 x$ for positive values of x. At the origin the value of μ is μ_0 aud we shall further assume that μ is constant and equal to zero for negative values of x.

Here ϵ_1 has been taken so small that its square and higher powers are neglected and terms of the first order of small quantities only have been retained. Motion of the fluid at a finite distance and at a great dlstance from the origin has been considered. We shall further suppose that u, v and w are small.

The general equations of motion[1] are

$$\rho_0 \frac{\mathrm{D}u}{\mathrm{D}t} = \left\{ \rho_0 X - \frac{\partial p}{\partial x} + \tfrac{1}{3}\mu\,\frac{\partial \theta}{\partial x} + \mu\nabla^2 u \right\} + \frac{\partial \mu}{\partial x}\left(2\frac{\partial u}{\partial x} - \frac{2}{3}\theta \right)$$

$$\rho_0 \frac{\mathrm{D}v}{\mathrm{D}t} = \left\{ \rho_0 Y - \frac{\partial p}{\partial y} + \tfrac{1}{3}\mu\,\frac{\partial \theta}{\partial y} + \mu\nabla^2 v \right\} + \frac{\partial \mu}{\partial x}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right)$$

$$\rho_0 \frac{\mathrm{D}w}{\mathrm{D}t} = \left\{ \rho_0 Z - \frac{\partial p}{\partial z} + \tfrac{1}{3}\mu\,\frac{\partial \theta}{\partial z} + \mu\nabla^2 w \right\} + \frac{\partial \mu}{\partial x}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right).$$

where $\qquad \theta \equiv \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial z}.$

Since u, v, w and ϵ_1 are all small, the general equations of motion, in the viscous medium, of sound waves become,

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + \frac{\mu_0+\epsilon_1 x}{3\rho_0}\frac{\partial \theta}{\partial x} + \frac{\mu_0+\epsilon_1 x}{\rho_0}\nabla^2 u$$

$$\frac{\partial v}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} + \frac{\mu_0+\epsilon_1 x}{3\rho_0}\frac{\partial \theta}{\partial y} + \frac{\mu_0+\epsilon_1 x}{\rho_0}\nabla^2 v \Bigg\} \quad \dots\dots(1)$$

$$\frac{\partial w}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} + \frac{\mu_0+\epsilon_1 x}{3\rho_0}\frac{\partial \theta}{\partial z} + \frac{\mu_0+\epsilon_1 x}{\rho_0}\nabla^2 w$$

If s denote the condensation we have, in addition, the equation of continuity

$$\frac{\partial s}{\partial t} = -\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right), \quad \dots\dots (2)$$

and the physical equation

$$p = p_0 + \rho_0 c^2 s, \quad \dots\dots (3)$$

where c is the velocity of sound in the absence of viscosity.

Now for motion at a finite distance from the origin, we have to neglect $\epsilon_1 x \frac{\partial \theta}{\partial x}, \epsilon_1 x \frac{\partial \theta}{\partial y}, \epsilon_1 x \frac{\partial \theta}{\partial z}, \epsilon_1 x \nabla^2 u, \epsilon_1 x \nabla^2 v$ and $\epsilon_1 x \nabla^2 w$ and this case has been already solved[2]. Again if we consider the motion at a great distance from the origin μ changes appreciably and we cannot neglect them.

Writing $\mu_0 + \epsilon_1 x = \epsilon_1 x_1$ and eliminating p and θ, we have

$$\frac{\partial u}{\partial t} = -c^2\frac{\partial s}{\partial x_1} + \frac{\epsilon_1 x_1}{\rho_0}\nabla^2 u - \tfrac{1}{3}\frac{\epsilon_1 x_1}{\rho_0}\frac{\partial}{\partial t}\frac{\partial s}{\partial x_1}$$

$$\frac{\partial v}{\partial t} = -c^2\frac{\partial s}{\partial y} + \frac{\epsilon_1 x_1}{\rho_0}\nabla^2 v - \tfrac{1}{3}\frac{\epsilon_1 x_1}{\rho_0}\frac{\partial}{\partial t}\frac{\partial s}{\partial y} \Bigg\} \quad \dots\dots (1.1).$$

$$\frac{\partial w}{\partial t} = -c^2\frac{\partial s}{\partial z} + \frac{\epsilon_1 x_1}{\rho_0}\nabla^2 w - \tfrac{1}{3}\frac{\epsilon_1 x_1}{\rho_0}\frac{\partial}{\partial t}\frac{\partial s}{\partial z}$$

where $\quad \frac{\partial u}{\partial x_1} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = -\frac{\partial s}{\partial t}$ and $\nabla^2 \equiv \frac{\partial^2}{\partial x_1{}^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$.

We assume that u, v, w and s all vary as $e^{\beta t}$. Thus we get

$$\beta u = \frac{\epsilon_1 x_1}{\rho_0}\nabla^2 u - \left(c^2 + \tfrac{1}{3}\frac{\epsilon_1 x_1}{\rho_0}\beta\right)\frac{\partial s}{\partial x_1}$$

$$\beta v = \frac{\epsilon_1 x_1}{\rho_0}\nabla^2 v - \left(c^2 + \tfrac{1}{3}\frac{\epsilon_1 x_1}{\rho_0}\beta\right)\frac{\partial s}{\partial y} \Bigg\} \quad \dots\dots (1.2).$$

$$\beta w = \frac{\epsilon_1 x_1}{\rho_0}\nabla^2 w - \left(c^2 + \tfrac{1}{3}\frac{\epsilon_1 x_1}{\rho_0}\beta\right)\frac{\partial s}{\partial z}$$

where
$$\frac{\partial u}{\partial x_1}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}= -\,\beta s, \qquad . \quad . \quad . \quad . \quad (2.1).$$
and β is a constant.

Differentiating both sides of the first, second and third equations of (2.1) with respect to x_1, y and z respectively and adding we have

$$\beta\left(\frac{\partial u}{\partial x_1}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)=\frac{\epsilon_1 x_1}{\rho_0}\nabla^2\left(\frac{\partial u}{\partial x_1}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)+\frac{\epsilon_1}{\rho_0}\nabla^2 u-\left(c^2+\tfrac{1}{3}\frac{\epsilon_1 x_1}{\rho_0}\beta\right)\nabla^2 s$$
$$-\tfrac{1}{3}\frac{\epsilon_1}{\rho_0}\beta\frac{\partial s}{\partial x_1}$$

or $\;-\beta^2 s= -\,\dfrac{\epsilon_1 x_1}{\rho_0}\beta\nabla^2 s-\left(c^2+\tfrac{1}{3}\dfrac{\epsilon_1 x_1}{\rho_0}\beta\right)\nabla^2 s-\tfrac{1}{3}\dfrac{\epsilon_1}{\rho_0}\beta\dfrac{\partial s}{\partial x_1}+\dfrac{\epsilon_1}{\rho_0}\nabla^2 u,\;$ by (2.1)

or $\qquad \beta^2 x_1 s=x_1\left(c^2+\tfrac{4}{3}\dfrac{\epsilon_1 x_1}{\rho_0}\beta\right)\nabla^2 s+\tfrac{1}{3}\dfrac{\epsilon_1 x_1}{\rho_0}\beta\dfrac{\partial s}{\partial x_1}-\dfrac{\epsilon_1 x_1}{\rho_0}\nabla^2 u,$

or $\qquad \beta^2 x_1 s=x_1\left(c^2+\tfrac{4}{3}\dfrac{\epsilon_1 x_1}{\rho_0}\beta\right)\nabla^2 s-c^2\dfrac{\partial s}{\partial x_1}-\beta\,u,\;$ . . . . . by (1.2)

or $\qquad \beta u=x_1\left(c^2+\tfrac{4}{3}\dfrac{\epsilon_1 x_1}{\rho_0}\beta\right)\nabla^2 s-c^2\dfrac{\partial s}{\partial x_1}-\beta^2 x_1 s\;$ . . . . . . (4)

Now $\qquad\left(\dfrac{\epsilon_1 x_1}{\rho_0}\nabla^2-\beta\right)u=\left(c^2+\tfrac{1}{3}\dfrac{\epsilon_1 x_1}{\rho_0}\beta\right)\dfrac{\partial s}{\partial x_1}$

or $\;\left(\dfrac{\epsilon_1 x_1}{\rho_0}\nabla^2-\beta\right)\left\{x_1\left(c^2+\tfrac{4}{3}\dfrac{\epsilon_1 x_1}{\rho_0}\beta\right)\nabla^2 s-c^2\dfrac{\partial s}{\partial x_1}-x_1\beta^2 s\right\}=\beta\left(c^2+\tfrac{1}{3}\dfrac{\epsilon_1 x_1}{\rho_0}\beta\right)\dfrac{\partial s}{\partial x_1},$ by(4)

or $\dfrac{\epsilon_1 x_1}{\rho_0}\Big[c^2\left\{x_1\nabla^2\left(\nabla^2 s\right)+2\dfrac{\partial}{\partial x_1}\nabla^2 s\right\}+\tfrac{4}{3}\dfrac{\epsilon_1}{\rho_0}\beta\left\{x_1{}^2\nabla^2\left(\nabla^2 s\right)+4\,x_1\dfrac{\partial}{\partial x_1}(\nabla^2 s)+2\nabla^2 s\right\}$

$\qquad\quad-c^2\dfrac{\partial}{\partial x_1}\nabla^2 s-\beta^2 x_1\nabla^2 s-2\beta^2\dfrac{\partial s}{\partial x_1}\Big]-\beta x_1\left(c^2+\tfrac{4}{3}\dfrac{\epsilon_1 x_1}{\rho_0}\beta\right)\nabla^2 s+c^2\beta\dfrac{\partial s}{\partial x_1}+x_1\beta^3 s$

$$=\beta c^2\dfrac{\partial s}{\partial x_1}+\tfrac{1}{3}\dfrac{\epsilon_1 x_1}{\rho_0}\beta^2\dfrac{\partial s}{\partial x_1}$$

or $\dfrac{\epsilon_1}{\rho_0}\Big[c^2\left\{x_1\nabla^2\left(\nabla^2 s\right)+\dfrac{\partial}{\partial x_1}\nabla^2 s\right\}+\dfrac{4\epsilon_1\beta}{3\rho_0}\left\{x_1{}^2\nabla^2\left(\nabla^2 s\right)+4\,x_1\dfrac{\partial}{\partial x_1}(\nabla^2 s)+2\nabla^2 s\right\}$

$\qquad\quad-\beta^2 x_1\nabla^2 s-2\beta^2\dfrac{\partial s}{\partial x_1}\Big]-\beta\left(c^2+\tfrac{4}{3}\dfrac{\epsilon_1 x_1}{\rho_0}\beta\right)\nabla^2 s+\beta^3 s=\tfrac{1}{3}\dfrac{\epsilon_1}{\rho_0}\beta^2\dfrac{\partial s}{\partial x_1}.$

or $c^2\left\{x_1\nabla^2\left(\nabla^2 s\right)+\dfrac{\partial}{\partial x_1}\nabla^2 s\right\}+\tfrac{4}{3}\dfrac{\beta}{\rho_1}\left\{x_1{}^2\nabla^2\left(\nabla^2 s\right)+4\,x_1\dfrac{\partial}{\partial x_1}(\nabla^2 s)+2\nabla^2 s\right\}$

$$-\beta\rho_1\left(c^2+\frac{4}{3}\frac{x_1}{\rho_1}\beta\right)\nabla^2 s + \beta^3\rho_1 s - \beta^2 x_1\nabla^2 s - 2\beta^2\frac{\partial s}{\partial x_1} = \frac{1}{3}\beta^2\frac{\partial s}{\partial x_1}, \quad \text{where } \rho_1 = \frac{\rho_0}{\epsilon_1}$$

or
$$c^2\left\{x_1\nabla^2(\nabla^2 s)+\frac{\partial}{\partial x_1}\nabla^2 s\right\}+\frac{4}{3}\frac{\beta}{\rho_1}\left\{x_1{}^2\nabla^2(\nabla^2 s)+4x_1\frac{\partial}{\partial x_1}(\nabla^2 s)+2\nabla^2 s\right\}-\beta^2 x_1\triangle^2 s$$

$$-\beta\rho_1\left(c^2+\frac{4}{3}\frac{x_1}{\rho_1}\beta\right)\nabla^2 s = \frac{7}{3}\beta^2\frac{\partial s}{\partial x_1}-\beta^3\rho_1 s, \quad . \quad . \quad . \quad . \quad (5)$$

To solve (5), let $s = \text{R.S}$ where S is a function of x_1 only and R that of y and z only, and R satisfies the equation $\nabla_1{}^2 R + kR = 0$, $\nabla_1{}^2$ being $\equiv \frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$. Thus Substituting the value of s in (5) we get the equation for S to be

$$c^2\left\{x_1\left(\frac{d^4 S}{dx_1{}^4}-2k\frac{d^2 S}{dx_1{}^2}+k^2 S\right)+\left(\frac{d^3 S}{dx_1{}^3}-k\frac{dS}{dx_1}\right)\right\}+\frac{4}{3}\frac{\beta}{\rho_1}\left\{x_1{}^2\left(\frac{d^4 S}{dx_1{}^4}-2k\frac{d^2 S}{dx_1{}^2}+k^2 S\right)\right.$$

$$+4x_1\left(\frac{d^3 S}{dx_1{}^3}-k\frac{dS}{dx_1}\right)+2\left(\frac{d^2 S}{dx_1{}^2}-kS\right)\right\}-\beta^2 x_1\left(\frac{d^2 S}{dx_1{}^2}-kS\right)-\beta\rho_1\left(c^2+\frac{4}{3}\frac{x_1}{\rho_1}\beta\right)\left(\frac{d^2 S}{dx_1{}^2}-kS\right)$$

$$-\frac{7}{3}\beta^2\frac{dS}{dx_1}+\rho_1\beta^3 S=0.$$

or
$$\left(c^2 x_1+\frac{4}{3}\frac{\beta}{\rho_1}x_1{}^2\right)\frac{d^4 S}{dx_1{}^4}+\left(c^2+\frac{16}{3}\frac{\beta}{\rho_1}x_1\right)\frac{d^3 S}{dx_1{}^3}+\left(-2kc^2 x_1-\frac{8k}{3}\frac{\beta}{\rho_1}x_1{}^2+\frac{8\beta}{3\rho_1}\right.$$

$$\left.-\beta^2 x_1-\beta\rho_1 c^2-\frac{4}{3}x_1\beta^2\right)\frac{d^2 S}{dx_1{}^2}+\left(-kc^2-\frac{16}{3}\frac{\beta}{\rho_1}kx_1-\frac{7}{3}\beta^2\right)\frac{dS}{dx_1}+\left(k^2 c^2 x_1+\frac{4\beta}{3\rho_1}k^2 x_1{}^2\right.$$

$$\left.-\frac{8k\beta}{3\rho_1}+k\beta^2 x_1+k\beta\rho_1 c^2+\frac{4}{3}k\beta^2 x_1+\beta^3\rho_1\right)S=0$$

or
$$x_1(3\rho_1 c^2+4\beta x_1)\frac{d^4 s}{dx_1{}^4}+(3\rho_1 c^2+16\beta x_1)\frac{d^3 s}{dx_1{}^3}+\{\beta(8-3\rho_1{}^2 c^2)-(6kc^2+7\beta^2)\rho_1 x_1$$

$$-8k\beta x_1{}^2\}\frac{d^2 S}{dx_1{}^2}-\{\rho_1(3kc^2+7\beta^2)+16\,k\beta x_1\}\frac{dS}{dx_1}+\{\beta(3\rho_1{}^2\beta^2+3\rho_1{}^2 c^2 k-8k)$$

$$+k\rho_1(7\beta^2+3kc^2)x_1+4\beta k^2 x_1{}^2\}S=0 \quad . \quad . \quad . \quad . \quad (6)$$

or $D_1 S = 0$ where D_1 stands for the operator of S. For its solution we construct an expression

$$V = \sum_{n=0}^{\infty} c_n x_1{}^{a+n}, \text{ so that}$$

$$D_1V = \Sigma c_0 \begin{bmatrix} \alpha(\alpha-1)(\alpha-2)(\alpha-3)(3\rho_1c^2+4\beta x_1){x_1}^{\alpha-3} + \alpha(\alpha-1)(\alpha-2)(3\rho_1c^2 \\ +16\beta x_1){x_1}^{\alpha-3} + \alpha(\alpha-1)\{\beta(8-3\rho_1^2c^2)-(6kc^2+7\beta^2)\rho_1x_1 \\ -8k\beta x_1^2\}{x_1}^{\alpha-2} - \{\rho_1(3kc^2+7\beta^2+16k\beta x_1)\alpha {x_1}^{\alpha-1} + \{\beta(3\rho_1^2\beta^2 \\ +3\rho_1^2c^2k-8k)+k\rho_1(7\beta^2+3kc^2)x_1+4\beta k^2x_1^2\}{x_1}^{\alpha} \end{bmatrix}$$

$$= \Sigma c_0 \begin{bmatrix} 3\rho_1c^2\alpha(\alpha-1)(\alpha-2)^2{x_1}^{\alpha-3} + \beta\alpha(\alpha-1)\{4\alpha(\alpha-1)-3\rho_1^2c^2\}{x_1}^{\alpha-2} \\ -\rho_1\alpha(A_1\alpha-A_2){x_1}^{\alpha-1} - \{8k\alpha(\alpha+1)-A_3\}\beta {x_1}^{\alpha}+A_4{x_1}^{\alpha+1} \\ +A_5{x_1}^{\alpha+2} \end{bmatrix}$$

where $A_1=6\ kc^2+7\beta^2$, $A_2=3kc^2$, $A_3=3\rho_1^2\beta^2+3\rho_1^2c^2k-8k$, $A_4=k\rho_1(3kc^2+7\beta^2)$ and $A_5=4k^2\beta$.

Thus $D_1 V=c_0 . 3\ \rho_1c^2 . \alpha(\alpha-1)(\alpha-2)^2 {x_1}^{\alpha-3}$ provided that

(i) $3\rho_1c^2(\alpha+1)\alpha(\alpha-1)^2 c_1+\beta\alpha(\alpha-1)\{4\alpha(\alpha-1)-3\rho_1^2c^2\}c_0=0$

or $3\rho_1c^2(\alpha+1)(\alpha-1)c_1+\beta\{4\alpha(\alpha-1)-3\rho_1^2c^2\}c_0=0$

(ii) $3\rho_1c^2(\alpha+2)(\alpha+1)\alpha.c_2+\beta(\alpha+1)\{4\alpha(\alpha+1)-3\rho_1^2c^2\}c_1-\rho(A_1\alpha+A_2)c_0=0$

(iii) $3\rho_1c^2(\alpha+3)(\alpha+2)(\alpha+1)^2 c_3+\beta(\alpha+2)(\alpha+1)\{4(\alpha+2)(\alpha+1)-3\rho_1^2c^2\}c_2$
$-\rho_1(\alpha+1)\{A_1(\alpha+1)-A_2\}c_1-\{8k\alpha(\alpha+1)-A_3\}\beta c_0=0$

(iv) $3\rho_1c^2(\alpha+4)(\alpha+3)(\alpha+2)^2c_4+\beta(\alpha+3)(\alpha+2)\{4(\alpha+3)(\alpha+2)-3\rho_1^2c^2\}c_3$
$-\rho_1(\alpha+2)\{A_1(\alpha+2)-A_2\}c_2-\{8k(\alpha+1)(\alpha+2)-A_3\}\beta c_1+A_4 c_0=0$

(v) $A_5c_{n-3}+A_4c_{n-2}-\{8k(\alpha+n-2)(\alpha+n-1)-A_3\}\beta c_{n-1}-\rho_1(\alpha+n)\{A_1(\alpha+n)$
$-A_2\}c_n+\beta(\alpha+n+1)(\alpha+n)\{4(\alpha+n+1)(\alpha+n)-3\rho_1^2c^2\}c_{n+1}$
$+3\rho_1c^2(\alpha+n+2)(\alpha+n+1)(\alpha+n)^2c_{n+2}=0$, for $n\geq 3$

or $3\rho_1c^2 . \dfrac{c_{n+2}}{c_{n-3}} + \dfrac{\beta\{4(\alpha+n+1)(\alpha+n)-3\rho_1^2c^2\}}{(\alpha+n+2)(\alpha+n)}\dfrac{c_{n+1}}{c_{n-3}} - \dfrac{\rho_1\{A_1(\alpha+n)-A_2\}.}{(\alpha+n+2)(\alpha+n+1)(\alpha+n)}\dfrac{c_n}{c_{n-3}}$

$-\dfrac{\beta\{8k(\alpha+n-2)(\alpha+n-1)-A_3\}}{(\alpha+n+2)(\alpha+n+1)(\alpha+n)^2}\dfrac{c_{n-1}}{c_{n-3}} + \dfrac{A_4}{(\alpha+n+2)(\alpha+n+1)(\alpha+n)^2}\dfrac{c_{n-2}}{c_{n-3}}$

$+\dfrac{A_5}{(\alpha+n+2)(\alpha+n+1)(\alpha+n)^2}=0$

If we suppose that $\dfrac{c_n}{c_{n-1}} \to l$ as $n \to \infty$, we have

$3\rho_1c^2l^5+4\beta\, l^4=0$. or $l^4(3\rho_1c^2l+4\beta)=0$.

Either $l=0$ or $-\dfrac{4\beta}{3\rho_1c^2}$.

F. 6

Now we discuss the convergence of the series corresponding to the limits given by $l=o$ and $-\dfrac{4\beta}{3\rho_1 c^2}$. Thus four cases arise,

(i) Corresponding to $l=o$, the series will be absolutely and uniformly convergent for all values of x_1.

(ii) If $|\,lx_1\,| = \left|\dfrac{4\beta x_1}{3\rho_1 c^2}\right| < 1$, the series will be absolutely convergent.

(iii) If $|\,lx_1\,| = \left|\dfrac{4\beta x_1}{3\rho_1 c^2}\right| > 1$, we get a divergent series.

(iv) If $|\,lx_1\,| = \left|\dfrac{4\beta x_1}{3\rho_1 c^2}\right| = 1$, the convergency or divergency of the

series will be doubtful.

So we turn to physical conditions.

$$-lx_1| = \left|\frac{4\beta x_1}{3\rho_1 c^2}\right| = \left|\frac{4\beta}{3c^2}\cdot\frac{\mu_0+\epsilon_1 x}{\rho_0}\right| = \left|\frac{4\beta}{3c^2}\cdot\frac{\mu}{\rho_0}\right|.$$

Considering air as the medium and temperature 20°C.[3], viscosity $=18\cdot1\times10^{-5}$ poise, $c=342.5$ metres/sec, $\rho_0=1.2928$ gm./litre, we find $|\,lx_1\,|$ is always less than one (for nearly all appreciably large values of β). Thus we find that the series corresponding to both the limits are absolutely convergent.

Now our indical equation is $\alpha\,(\alpha-1)\,(\alpha-2)^2=o$

$$ie.\ \alpha=2,\,2,\,1,\,o.$$

Therefore the four solutions are given by $S_1 = [V]_{\alpha=2}$,

$$S_2=\left[\frac{\partial V}{\partial\alpha}\right]_{\alpha=2}\quad S_3=\left[\frac{\partial^2 V}{\partial\alpha^2}\right]_{\alpha=1}\ \text{and}\ S_4=\left[\frac{\partial^3 V}{\partial\alpha^3}\right]_{\alpha=0}.$$

Now we turn to (1.2) and (2.1)

$$\left.\begin{aligned}
\beta u&=\frac{x_1}{\rho_1}\nabla^2 u -\left(c^2+\tfrac{1}{3}\frac{x_1}{\rho_1}\beta\right)\frac{\partial s}{\partial x_1}=\frac{x_1}{\rho_1}\nabla^2 u-\left(c^2+\tfrac{1}{3}\frac{x_1}{\rho_1}\beta\right)R\,\frac{dS}{dx_1}\\
\beta v&=\frac{x_1}{\rho_1}\nabla^2 v -\left(c^2+\tfrac{1}{3}\frac{x_1}{\rho_1}\beta\right)\frac{\partial s}{\partial y}=\frac{x_1}{\rho_1}\nabla^2 v-\left(c^2+\tfrac{1}{3}\frac{x_1}{\rho_1}\beta\right)S\,\frac{\partial R}{\partial y}\\
\beta w&=\frac{x_1}{\rho_1}\nabla^2 w -\left(c^2+\tfrac{1}{3}\frac{x_1}{\rho_1}\beta\right)\frac{\partial s}{\partial z}=\frac{x_1}{\rho_1}\nabla^2 w-\left(c^2+\tfrac{1}{3}\frac{x_1}{\rho_1}\beta\right)S\,\frac{\partial R}{\partial z}
\end{aligned}\right\} \quad\cdots\quad (1.3)$$

with $\dfrac{\partial u}{\partial x_1} + \dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial z} = -\beta s$.

Again from (4) $\beta u = x_1 \left(c^2 + \frac{4}{3}\dfrac{x_1}{\rho_1}\beta \right) \nabla^2 s - c^2 \dfrac{\partial s}{\partial x_1} - \beta^2 x_1 s.$

$$= R \left\{ x_1 \left(c^2 + \frac{4}{3}\dfrac{x_1}{\rho_1}\beta \right) \left(\dfrac{d^2 S}{dx_1^2} - kS \right) - c^2 \dfrac{dS}{dx_1} - \beta^2 x_1 S \right\}.$$

We Suppose $\qquad\qquad \beta v = \dfrac{\partial R}{\partial y} \cdot V_0$

and $\qquad\qquad\qquad \beta w = \dfrac{\partial R}{\partial z} \cdot V_0$ $\qquad\qquad\qquad\qquad$ (7)

where V_0 is a function of x_1 only,

so that $\qquad \beta\left(\dfrac{\partial u}{\partial x_1} + \dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial z} \right) = R \left\{ x_1\left(c^2 + \frac{4}{3}\dfrac{x_1}{\rho_1}\beta \right) \left(\dfrac{d^3 S}{dx_1^3} - k\dfrac{dS}{dx_1} \right) \right.$

$$\left. + \left(c^2 + \frac{8}{3}\dfrac{x_1}{\rho_1}\beta \right) \left(\dfrac{d^2 S}{dx_1^2} - kS \right) - c^2 \dfrac{d^2 S}{dx_1^2} - \beta^2 x_1 \dfrac{dS}{dx_1} - \beta^2 S \right\} - kRV_0$$

or $-\beta^2 S = x_1 \left(c^2 + \frac{4}{3}\dfrac{x_1}{\rho_1}\beta \right) \left(\dfrac{d^3 S}{dx_1^3} - k\dfrac{dS}{dx_1} \right) + \frac{8}{3}\dfrac{x_1}{\rho_1}\beta \left(\dfrac{d^2 S}{dx_1^2} - kS \right) - kc^2 S - \beta^2 x_1 \dfrac{dS}{dx_1}$

$$- \beta^2 S - kV_0$$

or $kV_0 = x_1 \left(c^2 + \frac{4}{3}\dfrac{x_1}{\rho_1}\beta \right) \dfrac{d^3 S}{dx_1^3} + \frac{8}{3}\dfrac{x_1}{\rho_1}\beta\dfrac{d^2 S}{dx_1^2} - x_1 \dfrac{dS}{dx_1}\left(\beta^2 + kc^2 + \frac{4}{3}k\dfrac{x_1}{\rho_1}\beta \right)$

$$- k\left(c^2 + \frac{8}{3}\dfrac{x_1}{\rho_1}\beta \right) S \qquad\qquad\qquad\qquad (I)$$

Again from the second or third equation of (1.3), we have, by using (7),

$$\beta V_0 = \dfrac{x_1}{\rho_1}\left(\dfrac{d^2 V_0}{dx_1^2} - kV_0 \right) - \beta\left(c^2 + \frac{1}{3}\dfrac{x_1}{\rho_1}\beta \right) S$$

or $\beta\rho_1 V_0 = x_1\left(\dfrac{d^2 V_0}{dx_1^2} - kV_0 \right) - \beta\rho_1\left(c^2 + \frac{1}{3}\dfrac{x_1}{\rho_1}\beta \right) S$

or $x_1\dfrac{d^2 V_0}{dx_1^2} - (\beta\rho_1 + kx_1) V_0 = \beta\rho_1\left(c^2 + \frac{1}{3}\dfrac{x_1}{\rho_1}\beta \right) S \qquad\qquad (II)$

In order that our supposition may be valid we must show that (I) and (II) tend to the same result.

From (I) $3\rho_1 kV_0 = x_1 (3\rho_1 c^2 + 4\beta x_1) \dfrac{d^3 S}{dx_1^3} + 8\beta x_1 \dfrac{d^2 S}{dx_1^2} - x_1\{3\rho_1(\beta^2 + kc^2) + 4k\beta x_1\}\dfrac{dS}{dx_1}$

$$- k (3\rho_1 c^2 + 8\beta x_1) S$$

$$3\rho_1 k \frac{dV_0}{dx_1} = x_1 \left(3\rho_1 c^2 + 4\beta x_1\right)\frac{d^4S}{dx_1^4} + \left(3\rho_1 c^2 + 8\beta x_1\right)\frac{d^3S}{dx_1^3} + 8\beta x_1 \frac{d^3S}{dx_1^3} + 8\beta \frac{d^2S}{dx_1^2}$$

$$- x_1\{3\rho_1(\beta^2 + kc^2) + 4k\beta x_1\}\frac{d^2S}{dx_1^2} - \{3\rho_1(\beta^2 + kc^2) + 8k\beta x_1\}\frac{dS}{dx_1} - k(3\rho_1 c^2 + 8\beta x_1)\frac{dS}{dx_1} - 8k\beta S$$

$$= \frac{d^2S}{dx_1^2}\left[4k\beta x_1^2 + 4\rho_1\beta^2 x_1 + 3\rho_1 kc^2 x_1 + 3\rho_1 c^2\beta\right] + \frac{dS}{dx_1}\left(4\rho_1\beta^2 - 3\rho_1 kc^2\right)$$

$$- \{\beta(3\rho_1^2\beta^2 + 3\rho_1^2 c^2 k) + k\rho_1(3kc^2 + 7\beta^2)x_1 + 4k^2\beta x_1^2\} S \quad \text{by using (6)}$$

$$3\rho_1 k \frac{d^2V_0}{dx_1^2} = \frac{d^3S}{dx_1^3}\{4k\beta x_1^2 + x_1(3\rho_1 kc^2 + 4\rho_1\beta^2) + 3\rho_1^2 c^2\beta\} + \frac{d^2S}{dx_1^2}(8k\beta x_1 + 3\rho_1 kc^2 + 8\rho_1\beta^2$$

$$- 3\rho_1 kc^2) - \{\beta(3\rho_1^2\beta^2 + 3\rho_1^2 c^2 k) + k\rho_1(3kc^2 + 7\beta^2)x_1 + 4k^2\beta x_1^2\}\frac{dS}{dx_1} - \{k\rho_1(3kc^2 + 7\beta^2)$$

$$+ 8k^2\beta x_1\} S$$

From these we get

$$3\rho_1 k\left[x_1\frac{d^2V_0}{dx_1^2} - (\beta\rho_1 + kx_1)V_0\right] = 3\rho_1 k.\ \beta\rho_1\left(c^2 + \frac{\beta}{3\rho_1}x_1\right)S.$$

Simplification leads to (II). Hence the justification.

Now we have to find the solution of

$$\left.\begin{array}{c} \beta u_1 = \dfrac{x_1}{\rho_1}\nabla^2 u_1 \\[2mm] \beta v_1 = \dfrac{x_1}{\rho_1}\nabla^2 v_1 \\[2mm] \beta w_1 = \dfrac{x_1}{\rho_1}\nabla^2 w_1 \\[2mm] \text{Subject to } \dfrac{\partial u_1}{\partial x_1} + \dfrac{\partial v_1}{\partial y} + \dfrac{\partial w_1}{\partial z} = 0- \end{array}\right\} \qquad \qquad (8)$$

$$\textit{i.e.}\quad (x_1\nabla^2 - k_0)(u_1, v_1, w_1) = 0 \quad \text{where } k_0 = \beta\rho_1$$

$$\text{with } \frac{\partial u_1}{\partial x_1} + \frac{\partial v_1}{\partial y} + \frac{\partial w_1}{\partial z} = 0.$$

and this case has been already solved [4].

Thus a complete solution can be obtained.

My best thanks are due to Prof A. C. Banerji who initiated me to this problem. My thanks are also due to principal M. Sengupta and Dr. S. S. Banerjee for their criticism and helpful suggestions.

REFERENCES

1. *Proc. Nat. Acad. Sci.* Vol. XII 1942 Page. 122.
2. Lamb. Hydrodynamics 6th. Edition, 1932. § 361.
3. Childs. *Physical Constants*, 1934, 25.
4. *Proc. Nat. Acad. Sci.* Vol. XII. 1942 Page 158.

A THEOREM FOR THE CONVERGENCE OF THE CONJUGATE SERIES OF A FOURIER SERIES

By S. S. Agarwal

Allahabad

Communicated by Dr. B. N. Prasad, Ph. D., D. Sc., F. N. A. Sc., F. N. I.

Department of Mathematics, University of Allahabad

1. Let $f(x)$ be a function which is integrable (L) over the interval $(-\pi, \pi)$ and defined outside by periodicity. Let the Fourier series of $f(x)$ be

$$(1.1) \quad \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

The conjugate series of this Fourier series is

$$(1.2) \quad \sum_{n=1}^{\infty} (b_n \cos nx - a_n \sin nx).$$

The conjugate function associated with (1.2) is

$$g(x) = \frac{1}{2\pi} \int_0^{\pi} \psi(t) \cot \frac{t}{2} \, dt.$$

$$(1.3) \qquad = \frac{1}{\pi} \int_0^{\infty} \frac{\psi(t)}{t} \, dt$$

where $\quad \psi(t) \qquad = f(x+t) - f(x-t).$

The object of this paper is to establish a test for the convergence of conjugate series, analogous to that given by Pollard for the Fourier series.[1]

We prove the following theorem :—

Theorem A. *At every point x at which the conjugate integral $g(x)$ exists the conjugate series (1.2) converges to $g(x)$ provided that*

---

[1] Pollard.

100

$$(1.4) \ldots \ldots \lim_{k \to \infty} \overline{\lim_{\epsilon \to 0}} \int_{k\epsilon}^{\delta} \left| \frac{|\psi(t+\epsilon)|}{t+\epsilon} - \frac{|\psi(t)|}{t} \right| dt = 0.$$

Further it will be shown that this test includes Young's[1] test for the convergence of the conjugate series.

I take this opportunity to express my best thanks to Dr. B. N. Prasad under whose guidance this paper has been written.

2.   We shall first prove two lemmas.

Lemma 1.   The existence of $g(x)$ implies that

$$(2.1) \quad \int_0^t \psi(t)\, dt = o(t).$$

This follows from Thomae's well-known theorem. Prasad[2] has generalized Thomae's result.

Lemma 2.   If k is a constant which tends to ∞ after n tends to ∞, then

$$(2.2) \quad \int_{\delta}^{k/n} \frac{\psi(t)}{t} \cos nt\, dt = 0\,(1) \text{ as } n \to \infty.$$

For $I = \int_0^{k/n} \frac{\psi(t)}{t} \cos nt\, dt$

$$= -\int_{\delta}^{k/n} \left\{ \frac{d}{dt} \int_t^{k/n} \frac{\psi(t)}{t}\, dt \right\} \cos nt\, dt$$

$$= -\left[\cos nt \int_t^{k/n} \frac{\psi(t)}{t}\, dt \right]_0^{k/n} - n \int_0^{k/n} \left\{ \int_t^{k/n} \frac{\psi(t)}{t}\, dt \right\} \sin nt\, dt.$$

Since $g(x)$ exists, the first term tends to zero by **Lemma 1.** and the second term in absolute value is equal to

$$k \cdot o\left(\frac{n}{k} \int_0^{k/n} dt \right) = k \left\{ o(1) \right\}$$

$$= o(1)$$

[1] Young.
[2] Prasad.

This proves the lemma.

3.   If S_n denotes the nth partial sum of (1.2) then,

$$(3.1) \quad S_n - g(x) = -\frac{1}{\pi} \int_0^\pi \frac{\psi(t)}{t} \cos nt \, dt + o(1)$$

$$= -\frac{1}{\pi} \left[\int_0^\eta + \int_\eta^\pi \right] + o(1)$$

$$= -\frac{1}{\pi} \int_0^\eta \frac{\psi(t)}{t} \cos nt \, dt + o(1),$$

by the Riemann-Lebesgue Theorem.

It follows from (2.2) that

$$\overline{\lim_{n\to\infty}} \int_0^\eta \frac{\psi(t)}{t} \cos nt \, dt = \overline{\lim_{n\to\infty}} \int_{k/n}^\eta \frac{\psi(t)}{t} \cos nt \, dt$$

$$= \lim_{k\to\infty} \overline{\lim_{n\to\infty}} \int_{k/n}^\eta \frac{\psi(t)}{t} \cos nt \, dt,$$

since a constant is its own limit.

This when (2.1) holds, (3.1) is equivalent to

$$S_n - g(x) = -\frac{1}{\pi} \lim_{k\to\infty} \overline{\lim_{n\to\infty}} \int_{k/n}^\eta \frac{\psi(t)}{t} \cos nt \, dt.$$

Hence in order to establish our theorem we have to prove

$$\lim_{k\to\infty} \overline{\lim_{n\to\infty}} \int_{k/n}^\eta \frac{\psi(t)}{t} \cos nt \, dt = o$$

Let
$$J = \int_{k/n}^\eta \frac{\psi(t)}{t} \cos nt \, dt$$

$$= \int_{k/v}^\eta \frac{\psi(t)}{t} \cos nt \, dt$$

where $$\frac{k}{n} = \frac{k}{\pi} \cdot \frac{\pi}{n} = k' \frac{\pi}{n} = k'v.$$

We can change the upper limit in J to $(\eta+v)$ and the lower limit to $k'v+v$, with error $o(1)$; the first change being trivial and the second justified by Riemann-Lebesgue theorem. If we do this and replace t by $t+v$, we obtain

$$J = - \int_{k/v}^{\eta} \frac{\psi(t+v)}{t+v} \cos nt \, dt.$$

Adding the two expressions for J we get

$$J = \tfrac{1}{2} \int_{k/n}^{\eta} \left\{ \frac{\psi(t)}{t} - \frac{\psi(t+v)}{t+v} \right\} \cos nt \, dt$$

$$= o(1) \text{ if } (1.4) \text{ is satisfied.}$$

The theorem is thus proved.

4. To prove that (1.4) also includes young's test for the convergence of the conjugate series, further suppose that

(4.1) $t \mid \psi(t) \mid$ is of bounded variation to the immediate right of $t=0$, and as $t \to 0$

$$\int_{0}^{t} \mid d\{ t \mid \psi(t) \mid \} \mid = O(t).$$

Let . $$\chi(t) = t \mid \psi(t) \mid$$

and $\theta(t) = \int_{0}^{t} \mid d\chi(u) \mid$

$$= \int_{0}^{t} \mid d\{ t\psi(t) \} \mid$$

On account of (4.1), δ can be so chosen that

$$\theta(t) \le A(t)$$

or $$\mid \chi(t) - \chi(o) \mid \le A(t)$$

But $\chi(o) = o$

$\therefore$ $$\mid \chi(t) \mid \le A(t).$$

F. 7

Hence

$$\frac{|\psi(t+\epsilon)|}{t+\epsilon} - \frac{|\psi(t)|}{t} = \frac{\chi(t+\epsilon)}{(t+\epsilon)^2} - \frac{\chi(t)}{t^2}$$

$$= \int_t^{t+\epsilon} d\,\frac{\chi(u)}{u^2}.$$

$$\therefore \left| \frac{|\psi(t+\epsilon)|}{t+\epsilon} - \frac{|\psi(t)|}{t} \right| \leq \int_t^{t+\epsilon} \left| d\,\frac{\chi(u)}{u^2} \right|$$

$$\leq \int_t^{t+\epsilon} d\,\frac{\theta(u)}{u^2} + 2\int_t^{t+\epsilon} \frac{|\chi(u)|}{u^3}\,du$$

$$= \left[\frac{\theta(u)}{u^2}\right]_t^{t+\epsilon} + 2\int_t^{t+\epsilon}\frac{\theta(u)}{u^3}\,du + 2\int_t^{t+\epsilon}\frac{|\chi(u)|}{u^3}\,du$$

$$= \left[\frac{\theta(u)}{u^2}\right]_t^{t+\epsilon} + 4\,A\left(\frac{1}{\epsilon} - \frac{1}{t+\epsilon}\right)$$

$$= \left[\frac{\theta(u)}{u^2} - \frac{4A}{t}\right]_t^{t+\epsilon}$$

So that

$$\int_{k\epsilon}^{\delta} \left\{ \left| \frac{|(t+\epsilon)|}{t+\epsilon} - \frac{|\psi(t)|}{t} \right| \right\} dt$$

$$\leq \int_{k\epsilon}^{\delta} \left[\frac{\theta(u)}{u^2} - \frac{4A}{u}\right]_t^{t+\epsilon} dt$$

$$= \int_{\delta}^{\delta+\epsilon} \left\{\frac{\theta(t)}{t^2} - \frac{uA}{u}\right\} dt - \int_{k\epsilon}^{(k+1)\epsilon}\left\{\frac{\theta(t)}{t^2} - \frac{4A}{t}\right\} dt$$

$$\leq \int_{\delta}^{\delta+\epsilon} \frac{\theta(t)}{t^2}\,dt + 4A\int_{k\epsilon}^{(kH)\epsilon} \frac{dt}{t^2}$$

$$\leq \; A \int\limits_{\delta}^{\delta+\epsilon} \frac{dt}{t} \; + \; 4A \int\limits_{k\epsilon}^{(k+1)\epsilon} \frac{dt}{t}$$

$$= \; A \log \frac{\delta+\epsilon}{\delta} \; + \; 4A \log\frac{k+1}{k},$$

which is annihilated by the operator $\varlimsup\limits_{\substack{k\to\infty \;\; \epsilon\to 0}}$

This proves the proposition.

REFERENCES

1. Pollard, S.—'Criteria for the convergence of a Fourier series' *Jour. Lond. Math. Soc.* 2,(1927), 255-262.

2. Young, W. H.—The convergence of a Fourier series and its allied series, *Proc. Lond. Math. Soc.*, (2) 10 (1911-12), 254-272.

3. Prasad, B. N.—Contribution á l' étude de la série conjuguée d'une série de Fourier. *Journal de Mathématiques*, 9, (1932), 153-205.

ON THE CESÀRO SUMMABILITY OF TRIGONOMETRIC SERIES

By M. L. Misra

Communicated by Dr. B. N. Prasad Ph. D., D. Sc., F. N. A. Sc., F. N. I.,

Department of Mathematics University of Allahabad.

1. Supose that $f(x)$ is a function, which is integrable in the sense of Lebesgue over the interval $(-\pi, \pi)$ and is defined outside this interval by periodicity. Let the Fourier series associated with $f(x)$ be

$$(1.1) \qquad \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos n x + b_n \sin n x).$$

The 'conjugate' series of this Fourier series is

$$(1.2) \qquad \sum_{n=1}^{\infty} (b_n \cos n x - a_n \sin n x).$$

The 'conjugate' function associated with (1.2) is

$$(1.3) \qquad g(x) = \frac{1}{2\pi} \int_0^{\pi} \{ f(x+t) - f(x-t) \} \cot \tfrac{1}{2} t \, dt,$$

$$(1.4) \qquad = \frac{1}{\pi} \int_0^{\infty} \frac{\psi(t)}{t} \, dt,$$

where

$$\psi(t) = f(x+t) - f(x-t),$$

the integrals being Cauchy integrals.

In this paper, we investigate the summability (C, k), where $k = n + a$, n being a positive integer and $o \leq a < 1$, of the conjugate series (1.2) by a general method which reduces the discussion of the summability $(C, n+a)$ to that of ordinary convergence or summability (C, a) according as a is equal to or greater than zero. In the former case we deduce a number of theorems on the lines of Prasad* by applying the known convergence

---

* Prasad, **12.**

criteria of the conjugate series of a Fourier series. In the second part of the paper we give the corresponding investigation for the Fourier series. It may be mentioned that there is a certain amount of overlapping between some of the results of this paper and those of Bosanquet, Verblunsky and Prasad (Theorems 3, 4 and 8). But whereas the technique adopted by Bosanquet and Verblunsky, although powerful, is of a rather difficult and complicated character, the analysis employed here has the advantage of simplicity and directness.

I am much indebted to Dr. B. N. Prasad for his kind interest and advice in the preparation of this paper.

2. We write

$$\gamma_{1+k}(t) = \int_0^1 (1-u)^k \cos t u \, du = \frac{\Gamma(k+1)}{t^{k+1}} C_{k+1}(t),$$

the function $C_{k+1}(t)$ being the case $p=k+1$ of Young's fuction*

$$C_p(t) = \sum_{r=0}^{\infty} (-1)^r \frac{t^{p+2r}}{\Gamma(p+2r+1)}.$$

It is well-known that †

(2.1) $$| r_k(t) | \leq \frac{A}{|(t)|^{\alpha}},$$

for all t and $k > 0$, where $\alpha = \min(k, 2)$ and A is a constant and

(2.2) $$| r'_k(t) | \leq \frac{A}{|t|^{\beta}},$$

for $k \geq 1$, where $\beta = \min(k, 3)$.

Further we write

$$\sigma_{1+k}(t) = \int_0^1 (1-u)^k . \sin t u \, du$$

(2.3) $$= \frac{t \, r_{2+k}(t)}{1+k} = \frac{1-k \, r_k(t)}{t}.$$

Also

(2.4) $$\sigma_{1+k}(t) \to 0, \text{ as } t \to 0, \text{ and } \sigma_{1+k}(t) = O\left(\frac{1}{t}\right), \text{ as } t \to \infty.$$

* Young, **17**, 160.
† Grimshaw, **2**; Pollard, **9**.

Further $r_k(t)$ and its derivatives are uniformly bounded for all t.

3.   We begin by giving a direct proof of a special case of Hardy and Littlewood's theorem* in the form of

Lemma 1.   *The allied series* (1·2) *is summable* (C, k), $k > 0$, *to the conjugate function* $g(x)$ *for all values of* x *for which the integral* (1 4) *exists, provided that*

$$\psi(t) = \frac{1}{t} \int_0^t |\psi(u)| \, du = O(1).$$

A particular of this lemma is Prasad's theorem† namely that if $f(x)$ is bounded, the conjugate series is summable (C, δ), $\delta > 0$, to $g(x)$.

As shown by Prasad and Verblunsky‡, the condition that the conjugate series may be summable (C, k) to S at $t = x$ is that

(3.1)
$$\lim_{\omega \to \infty} \frac{\omega}{\pi} \int_0^\infty \psi(t) \, \sigma_{1+k}(\omega t) \, dt = S.$$

If we take

$$S = \frac{1}{\pi} \int_0^\infty \frac{\psi(t)}{t} \, dt,$$

then (3.1) reduces to

(3.2)
$$\lim_{\omega \to \infty} \int_0^\infty \frac{\psi(t)}{t} \, r_k(\omega t) \, dt = 0,$$

by (2.3).   Thus to prove the lemma, we have to show that (3.2) holds if

$$\Psi(t) = O(1)$$

Now

$$\int_0^\infty \frac{\psi(t)}{t} \, r_k(\omega t) \, dt = \left(\int_0^{\lambda/\omega} + \int_{\lambda/\omega}^\eta + \int_\eta^\infty \right) \frac{\psi(t)}{t} \, r_k(\omega t) \, dt$$

$$= I + J + K,$$

---

* Hardy and Littlewood, **4**, Theorem B (ii).
† Prasad, **10**, 274.
‡ Prasad, **10**, 277. ; Verblunsky, **16**, 400.

say. Integrating by parts, get

$$I = -\left[r_k(\varpi t) \int\limits_t^{\lambda/\omega} \frac{\psi(t)}{t}\, dt \right]_0^{\lambda/\omega} + \omega \int\limits_0^{\lambda/\omega} \left\{ \int\limits_t^{\lambda/\omega} \frac{\psi(t)}{t}\, dt \right\} r'_k(\varpi t)\, dt$$

$$= r_k(0) \int\limits_0^{\lambda/\omega} \frac{\psi(t)}{t}\, dt + o\left(\omega \int\limits_0^{\lambda/\omega} dt \right)$$

$$= o\,(1),$$

as $\omega \to \infty$, since the integral (1.4) exists and $r_k'(\varpi t)$ is bounded in $(0, \lambda/\omega)$ for $k > 0$. Also by (2.1)

$$|J| \leq A \int\limits_{\lambda/\omega}^{\eta} \frac{|\psi(t)|}{t} \cdot \frac{dt}{(\omega t)^a}\ ,\ a = \min(k, 2) > 0,$$

$$= \frac{A}{\omega^a} \left[\frac{\psi(t)}{t^a} \right]_{\lambda/\omega}^{\eta} + \frac{A\,(a+1)}{\omega^a} \int\limits_{\lambda/\omega}^{\eta} \frac{\psi(t)}{t^{1+a}}\, dt$$

$$< \frac{A}{\omega^a} + \frac{\psi(\eta)}{\eta^a} + \frac{A}{\omega^a}\, \psi \left(\int\limits_{\lambda/\omega}^{\eta} \frac{dt}{t^{1+a}} \right)$$

$$= O\left(\frac{1}{\omega^a} \right) + O\left(\frac{1}{\lambda^a} \right) = o\,(1),$$

if $\lambda \to \infty$ after $\omega \to \infty$.

Further,

$$|K| = \left| \int\limits_\eta^\infty \frac{\psi(t)}{t}\, r_k(\varpi t)\, dt \right| < \frac{A}{\omega^k} \int\limits_\eta^\infty \frac{|\psi(t)|}{t^{1+k}}\, dt$$

$$\leq \frac{A}{\omega^k} \int\limits_0^\pi |\psi(t)| \left\{ \frac{1}{\eta^{1+k}} + \frac{1}{1^{1+k}} - \frac{1}{2^{1+k}} + \ldots \right\} dt$$

$$= O\left(\frac{1}{\omega^k} \right) = o\,(1),$$

as $\omega \to \infty$. This proves the lemma for $k > 0$.

If however $k > 1$, we have

$$J = \int_{\lambda/\omega}^{\eta} \frac{\psi(t)}{t} r_k(\omega t)\, dt$$

$$= -\left[r_k(\omega t) \int_t^{\eta} \frac{\psi(t)}{t}\, dt \right]_{\lambda/\omega}^{\eta} + \omega \int_{\lambda/\omega}^{\eta} \left\{ \int_t^{\eta} \frac{\psi(t)}{t}\, dt \right\} r'_k(\omega t)\, dt$$

$$= J_1 + J_2,$$

say. Now

$$|J_1| = \frac{1}{\lambda^a} \left| \int_{\lambda/\omega}^{\eta} \frac{\psi(t)}{t}\, dt \right|, a = \min(k, 2) > 1,$$

$$= o(1).$$

if $\lambda \to \infty$ after $\omega \to \infty$. Also if M be the upper bound of $\left| \int_0^{\eta} \frac{\psi(t)}{t}\, dt \right|$ in (o, η),

then

$$|J_2| \leq M\,\omega \int_{\lambda/\omega}^{\eta} |r'_k(\omega t)|\, dt$$

$$< \frac{M\omega}{\omega^{\beta}} \int_{\lambda/\omega}^{\eta} \frac{dt}{t^{\beta}}, \beta = \min(k, 3) > 1,$$

$$< \frac{M}{(\beta-1)} \cdot \frac{1}{\lambda^{\beta-1}} = o(1),$$

if $\lambda \to \infty$ after $\omega \to \infty$, β being greater than 1. Thus

$$J = o(1).$$

Also

$$I = o(1), K = o(1),$$

as before. Hence we get Paley's result* that if $k > 1$, the conjugate series is always summable (C, k) to

$$\frac{1}{\pi} \int_0^{\infty} \frac{\psi(t)}{t}\, dt,$$

wherever this integral exists.

---

* Paley, **8**, Theorem 2 with $a = 1$.

4. In the lemma 1 above, we have supposed that the conjugate integral (1·4), exists. We shall now remove this restriction and assume instead, that the conjugate integral (1·4) exists (C, r), that is, if

$$\chi(\epsilon) = \int_{\epsilon}^{X} \frac{\psi(t)}{t}\, dt,$$

then

$$\chi_r(\epsilon) = \frac{1}{\epsilon} \int_0^{\epsilon} \frac{d\epsilon_1}{\epsilon_1} \int_0^{\epsilon_1} \frac{d\epsilon_2}{\epsilon_2} \cdots \cdots \int_0^{\epsilon_{r-1}} \chi(\epsilon_r)\, d\epsilon_r$$

tends to a limit when $\epsilon \to 0$ and $X \to \infty$, the case $r = 0$ implying convergence in the ordinary sense. Let

$$\psi_0(t) = \psi(t) = f(x+t) - f(x-t),$$

$$\psi_1(t) = \frac{1}{t} \int_0^t \psi(t)\, dt, \quad \psi_2(t) = \frac{1}{t} \int_0^t \psi_1(t)\, dt,$$

and generally

$$\psi_n(t) = \frac{1}{t} \int_0^t \psi_{n-1}(t)\, dt,$$

n being a positive integer. We shall now prove

Theorem 1. *The conjugate series corresponding to $\psi(t)$ will be summable* (C, k) *to S if the conjugate series for $\psi_n(t)$ is summable* (C, k-n) *to S, $k \geq n$.*

5. **Proof of Theorem 1.** As before, the condition that the conjugate series be summable (C, k) to S is that

(5.1) $$\lim_{\omega \to \infty} \frac{\omega}{\pi} \int_0^{\infty} \psi(t)\, \sigma_{1+k}(\omega t)\, dt = S.$$

Now

$$\frac{\omega}{\pi} \int_0^{\infty} \psi(t)\, \sigma_{1+k}(wt)\, dt = \frac{1}{\pi} \left[wt\, \psi_1(t)\, \sigma_{1+k}(wt) \right]_0^{\infty} - \frac{w^2}{\pi} \int_0^{\infty} t\, \psi_1(t)\, \sigma'_{1+k}(wt)\, dt.$$

The terms within the square brackets tend to zero as

F. 8

$$\psi_1(t) = \frac{1}{t}\int_0^t \psi(t)\,dt = O\left(\frac{1}{t}\right), \text{ as } t \to 0,$$

$$= O\left(\frac{1}{t}\right), \text{ as } t \to \infty,$$

and by (2.4).　Hence

$$\frac{\omega}{\pi}\int_0^\infty \psi(t)\,\sigma_{1+k}(wt)\,dt = -\frac{\omega^2}{\pi}\int_0^\infty t\,\psi_1(t)\,\sigma'_{1+k}(wt)\,dt$$

$$= -\frac{\omega k}{\pi}\int_0^\infty \psi_1(t)\,\sigma_k(wt)\,dt + \frac{\omega(k+1)}{\pi}\int_0^\infty \psi_1\,\sigma_{1+k}(wt)\,dt,$$

since

$$t\,\sigma'_{1+k}(t) = k\,\sigma_k(t) - (k+1)\,\sigma_{1+k}(t).$$

Or

$$\frac{\omega}{\pi}\int_0^\infty \psi(t)\,\sigma_{1+k}(wt)\,dt - S = -k\left[\frac{\omega}{\pi}\int_0^\infty \psi_1(t)\sigma_k(wt)\,dt - S\right] +$$

$$(k+1)\left[\frac{\omega}{\pi}\int_0^\infty \psi_1(t)\,\sigma_{1+k}(wt)\,dt - S\right].$$

Or

$$I = -k\,P + (k+1)\,Q,$$

say. Now as $\omega \to \infty$, $I = o(1)$ is the condition that the conjugate series corresponding to $\psi(t)$ is summable (C, k) to S ; $P = o(1)$ is the condition that the conjugate series corresponding to $\psi_1(t)$ is summable $(C, k\text{-}1)$ to S and $Q = o(1)$ is the condition for the summability (C, k) to S of the conjugate series corresponding to $\psi_1(t)$. From the theory of summability it follows that, when $P = o(1)$, we have $Q = o(1)$ and therefore $I = o(1)$. Hence the condition that the conjugate series for $\psi(t)$ may be summable (C, k) is that the conjugate series for $\psi_1(t)$ be summable $(C, k\text{-}1)$, $k \geq 1$. By repeated application of this result, we obtain the theorem.

　　I. If we take $k = n + a$, where n is a positive integer or zero and $o \leq a < 1$, then the Theorem I becomes

Theorem 1 A. *The conjugate series corresponding to* $\psi(t)$ *will be* summable (C, $n + a$), n being a positive *integer or zero and* $0 \leq a < 1$, *to* S *if the conjugate series for* $\psi_n(t)$ *is summable* (C, a) *to* S, *that is if*

(5.2).
$$\lim_{\omega \to \infty} \frac{\omega}{\pi} \int_0^\infty \psi_n(t)\, \sigma_{1+k}(wt)\, dt = \text{S}.$$

6. We shall now examine the cases $a > 0$ and $a = 0$.

Case 1. If $a > 0$, we shall prove the theorem.

Theorem 2. *The conjugate series is summable* (C, k) *to the integral.*

$$\frac{1}{\pi} \int_0^\infty \frac{\psi(t)}{t}\, dt \; (\text{C}, n),$$

where $k = n + a$, *n being a positive integer including zero and* $0 < a < 1$, *at every point where the integral exists provided that*

$$\frac{1}{t} \int_0^t |\psi_n(u)|\, du = \text{O}(1).$$

The lemma 1 above is the case $n = 0$ of the theorem 2.

To prove this theorem we require the lemma.

Lemma **2.** *If the integral*

$$\int_0^\infty \frac{\psi(t)}{t}\, dt \; (\text{C}, r)$$

exists, r being a positive integer, then the integral

$$\int_0^\infty \frac{\psi_r(t)}{t}\, dt$$

also exists and is equal to it.

This follows from the lemma ζ of hardy and Littlewood* by putting $a = \infty$.

———————
     * Hardy and Littlewood, **3**, 222.

Proof of Theorem 2. If we take

$$S = \frac{1}{\pi}\int_0^\infty \frac{\psi_n(t)}{t}\ dt = \frac{1}{\pi}\int_0^\infty \frac{\psi(t)}{t}\ dt \ (C, n)$$

by the lemma 2, then by the lemma 1, (5.2) and (5.1) would hold, provided that

$$\frac{1}{t}\int_0^t |\ \psi_n(u)\ |\ du = O\ (1).$$

This proves the theorem 2. We also observe that if $k > n + 1$, the series is always summable to

$$\frac{1}{\pi}\int_0^\infty \frac{\psi(t)}{t}\ dt \ (C, n),$$

which is Paley's result.*

We shall next prove

Theorem 3. *At a point at which*

$$\frac{1}{t}\int_0^t \psi_n(u)\ du = 0\ (1), \frac{1}{t}\int_0^t |\ \psi_n(u)\ |\ du = O\ (1),$$

a necessary and sufficient condition that the conjugate series be summable (C, $n + a$) *to* S *is that the conjugate integral* (1.4) *should converge* (C; n) *to* S.

We require the following lemmas :—

Lmma 3. *If* $\psi_r(t) = o\ (1)$ *and* $\int_0^\infty \frac{\psi(t)}{t}\ dt$ (C, r) *exists, then*

$$\int_0^\infty \frac{\psi(t)}{t}\ dt \ (C,\ r - 1)\ also\ exists.$$

This easy to show with the help of lemma 2 and the integration by parts.

Lemma 4. *If* $\psi_{n+1}(t) = o\ (1)$ *and if the conjugate integral exists in some Cesàro sense, then it exists* (C, n).

---

* Paley, **8**, Theorem 2 with $a = n$.

For by the lemma 2,

$$\frac{1}{\pi}\int_0^\infty \frac{\psi(t)}{t}\, dt\ (C,\ n+1) = \frac{1}{\pi}\int_0^\infty \frac{\psi_{n+1}(t)}{t}\, dt.$$

Now if the conjugate integral exists in some Cesàro sense, then the integral

$$\frac{1}{\pi}\int_0^\infty \frac{\psi_{n+1}(t)}{t}\, dt$$

either converges or exists in some Cesàro sense.   But

$$\frac{1}{t}\psi_{n+1}(t) = o\left(\frac{1}{t}\right).$$

Hence by the integral analogue of Cesaro-Tauber theorem for summable series, the integral

$$\frac{1}{\pi}\int_0^\infty \frac{\psi_{n+1}(t)}{t}\, dt = \frac{1}{\pi}\int_0^\infty \frac{\psi(t)}{t}\, dt\ (C,\ n+1)$$

exists.   Also

$$\psi_{n+1}(t) = o\,(1).$$

Therefore by the lemma 3, the conjugate integral exists $(C,\ n)$.

Coming now to the proof of Theorem 3, we see that the condition is necessary because if the series be summable $(C,\ n)$ it is summable (A) and hence the conjugate integral exists in some Cesaro sense by a result proved by the author* and then by the lemma 4, it exists $(C,\ n)$.   The sufficiency of the condition follows from the theorem 2.

Theorem 2 completes the theorem VIII of Verblunsky.†   A similar theorem has been given by Bosanquet.‡

If we assume only the second condition of the theorem 3, then we shall get

---

* Misra, **7** (Theorem A).

† Verblunsky, **16**, with $p = n$ (integer).   His functions $\chi(t)$ are our $\psi(t)$.

‡ Bosanquet, **1**, 28

Theorem 3 A. *At a point at which*

$$\frac{1}{t}\int_0^t |\, \psi_n\,(u)\,|\; du = O\,(1),$$

a necessary and sufficient condition that the conjugate series be summable (C, $n+1+a$), $a>0$, *to the sum* S *is that the coujugate integral* (1.4) *should converge* (C, $n+1$) *to* S.

This theorem follows from theorem 2 and the following lemmas :—

Lemma 5. *If*

$$\frac{1}{t}\int_0^t |\, \psi_n\,(u)\,|\; du = O\,(1),$$

and if the conjugate integral exists in some Cesàro sense, then it exists (C, $n+1$).

For we have

$$\psi_{n+1}\,(t) = \frac{1}{t}\int_0^t \psi_n\,(u)\; du = O\,(1)$$

and so the proof is as in the lemma 4.

Lemma 6. *If*

$$\frac{1}{t}\int_0^t |\, \psi_n\,(u)\,|\; du = O\,(1),$$

then

$$\frac{1}{t}\int_0^t |\, \psi_{n+1}\,(u)\,|\; du = O\,(1).$$

This is due to Verblunsky.*

7. Case 2. If $a=0$, $k=n$ (a positive integer), then the conditon (5.2) reduces to

$$\lim_{\omega\to\infty} \frac{\omega}{\pi}\int_0^\infty \psi_n\,(t)\,\sigma_1\,(wt)\; dt = S.$$

---

* Verblunsky, **15**, Theorem IX B with $\delta=1$.

Or

(7.1)
$$\lim_{\omega \to \infty} \frac{1}{\pi} \int_0^\infty \psi_n(t) \frac{1 - \cos \omega t}{t} \, dt = S.$$

If we take

$$S = \frac{1}{\pi} \int_0^\infty \frac{\psi_n(t)}{t} \, dt = \frac{1}{\pi} \int_0^\infty \frac{\psi(t)}{t} \, dt \ (C, n),$$

then (7.1) becomes

(7.2)
$$\lim_{\omega \to \infty} \int_0^\infty \psi_n(t) \frac{\cos \omega t}{t} \, dt = 0,$$

and we obtain, from the theorem 1 A, the following theorem :—

Theorem 4. *The conjugate series for* $\psi(t)$ *will be summable* (C, n), n *being a positive integer or zero, to*

$$S = \frac{1}{\pi} \int_0^\infty \frac{\psi(t)}{t} \, dt \ (C, n)$$

provided that the conjugate series for $\psi_n(t)$ *converges to* S, *that is if,*

(7.2)
$$\lim_{\omega \to \infty} \int_0^\infty \psi_n(t) \frac{\cos \omega t}{t} \, dt = 0.$$

If $n = 0$, we have the well-known theorem for the convergence of the conjugate series.*

This reduces the summability (C, n) of the conjugate series corresponding to $\psi(t)$ to the ordinary convergence of the conjugate series for $\psi_n(t)$. If we now apply to (7.2) the well-known standard criteria for convergence of the conjugate series, we shall have as many tests for summability (C, n). This we get the following tests :—

By the application of Pringsheim's test. †

Theorem 5 (a). *The conjugate series is summable* (C, n) *to*

---

* Hobson, **5**, 694.
† Pringsheim, **17**, 87.

$$\frac{1}{\pi}\int_0^\infty \frac{\psi(t)}{t}\,dt \ (C, n),$$

if the integral

$$\int_0^\delta \left| \frac{\psi_n(t)}{t} \right| dt$$

exists.

By the application of Young's tests*

Theorem 5 (b).   *The conjugate series is summable* (C, n) *to*

$$\frac{1}{\pi}\int_0^\infty \frac{\psi(t)}{t}\,dt \ (C, n),$$

at a point, if a neighbourhood can be found in which $\psi_n(t)$ *is of bounded variation or more generally*

$$\frac{1}{t}\int_0^\infty \psi_n(t)\,dt$$

is of bounded variation.

By the application of Prasad's test†

Theorem 5 (c).   *The conjugate series is summable* (C, n) *to*

$$\frac{1}{\pi}\int_0^\infty \frac{\psi(t)}{t}\,dt \ (C, n),$$

provided that

$$\psi_n(t) = \frac{1}{t}\int_0^t \psi_{n-1}(t)\,dt$$

where

$$\int_0^t | \psi_{n-1}(t) |\,dt = O(t).$$

By the application of a test given by the author ‡—

* Hobson, **5**, 696.

† Prasad, **11**, theorem 2. ;   ‡ Misra, **6**, 213. ;

Theorem 5 (d).   *The conjugate series is summable* (C, n) *to*

$$\frac{1}{\pi}\int_0^\infty \frac{\psi(t)}{t}\, dt\ (t)\ (C, n),$$

provided that _

$$\lim_{\epsilon \to 0}\ \int_\epsilon^\delta \left| \frac{\psi_n(t)}{t} - \frac{\psi_n(t+2\epsilon)}{t+2\epsilon} \right| dt = 0,\ \delta > \epsilon.$$

The theorem 5 (d) includes the theorems 5 (a), 5 (b) and 5 (c). As shown elesewhere*, Prasad's generalised conjugate functions $g(x)\, g_r(x)$ are respectively equal to the integrals

$$\frac{1}{\pi}\int_0^\infty \frac{\psi(t)}{t}\ (C, 1)\ \text{and}\ \frac{1}{\pi}\int_0^\infty \frac{\psi(t)}{t}\, dt\ (C, r).$$

The theorem 4 of Prasad is a particular case $n=1$ of the theorem 2 above; also the theorem 5 of Prasad ‡ is the theorem 4 above. Our theorems 5 (a), 5 (b) and 5 (c) correspond to the theorem 6 of Prasad. †

8. Hence-forward, we shall give the corresponding investigation for the Fourier series.

It is know that a necessaay and sufficient condition for the Fourier series of an integrable and periodic function $f(t)$ to be summable (C, k), $k > 0$, to the sum S, at the point $t=x$ is

(8.1) $$\lim_{\omega \to \infty} \omega \int_0^\eta \phi(t)\, r_{1+k}(\omega t)\, dt = 0,$$

where

$$\phi(t) = f(x+t) + f(x-t) - 2\, S.$$

Let

$$\phi_0(t) = \phi(t),\ \phi_1(t) = \frac{1}{t}\int_0^t \phi(t)\, dt,\ \phi_r(t) = \frac{1}{t}\int_0^t \phi_{r-1}(t)\, dt,$$

* Misra, **7**. ; ‡ Prasad, **11**, 160, 162.
† Prasad, **11**, 178, 188, 196.

F. 9

r being a positive integer. we say that

$$\phi\,(t) = o\,(1)\,(C,\,r)\ \text{if}\ \phi_r\,(t) \to o\ \text{as}\ t \to o.$$

Verblumsky* has proved the following result :—

Lemma 7.   If

$$\frac{1}{t}\int_0^t \phi\,(t)\,dt = 0\,(1),\ \frac{1}{t}\int_0^t |\,\phi\,(t)\,|\,dt = O\,(1),$$

then

$$\lim_{\omega \to \infty}\omega \int_0^{\eta} \phi\,(t)\,r_{1+k}\,(\omega t)\,dt = o,\ k > 0.$$

We have, by integration by parts, and using (2·1),

$$\lim_{\omega \to \infty}\omega \int_0^{\eta} \phi\,(t)\,r_{1+k}\,(\omega t)\,dt = \lim_{\omega \to \infty}\left[\omega\eta\phi_1\,(\eta)\,r_{1+k}\,(\omega\eta) - \omega^2 \int_0^{\eta}\phi_1\,(t)\,t\,r'_{1+k}\,(\omega t)\,dt\right]$$

$$= -\lim_{\omega \to \infty}\omega^2 \int_0^{\eta} t\,\phi_1\,(t)\,r'_{1+k}\,(\omega t)\,dt$$

$$= -\lim_{\omega \to \infty}k\omega \int_0^{\eta}\phi_1\,r_k\,(\omega t)\,dt + \lim_{\omega \to \infty}\omega\,(k+1)\int_0^{\eta}\phi_1\,r_{1+k}\,(\omega t)\,dt,$$

as

$$t\,r'_{1+k}\,(t) = k\,r_k\,(t) - (k+1)\,r_{1+k}\,(t).$$

or

$$\text{I} = -\,k\,\text{P} + (k+1)\,\text{Q},$$

say.

Now I=o is the condition that the Fourier series for $\phi\,(t)$ is summable (C, k) to S ; P=o is the condition that the Fourier series for $\phi_1\,(t)$ is summable $(C, k-1)$ to S and Q=o is the condition that the Fourier series for $\phi_1\,(t)$ summable (C, k) to S. We know that, when P=o, we have

$$\text{Q}=o\quad\text{and hence I} = o.$$

Thus we get

---

* Verblunsky, **16**, Lemma 1 .

Theorm 6. *The Fourier series corresponding to $\phi(t)$ will be summable (C, k) to the sum S if the Fourier series for $\phi_1(t)$ is summable (C, k − 1) to S, k ≥ 1.*

This generalises Theorem 1 of Prasad.*

If $k = n + \alpha$, where n = a positive integer and $0 \leq \alpha < 1$, then by repeated application of the theorem 6 we get

Theorem 7. *The Fourier series corresponding $\phi(t)$ will be summable (C, n + α) to the sum S if the Fourier series for $\phi_n(t)$ is summable (C, n + α) to S i.e. if*

$$(8.2). \quad \lim_{\omega \to \infty} \omega \int_0^\eta \phi_n(t)\, r_{1+\alpha}(\omega t)\, dt = 0.$$

9. Case I. If $\alpha > 0$, then lemma 7 gives

Theorem 8. *The Fourier series for $\phi(t)$ will be summable (C, n + α) to S if*

$$\frac{1}{t}\int_0^t \phi_n(t)\, dt = o\,(1), \quad \frac{1}{t}\int_0^t |\phi_n(t)|\, dt = O\,(1),$$

n being a positive integer and $0 < \alpha < 1$.

This is Verblunsky's† theorem I for $p = n$. We can also, as in the case of conjugate series, establish theorems analogous to the theorems II and III of Verblunsky.‡

10. Case II. If $\alpha = 0$, $k = n$, a positive integer, then the theorem 7 becomes.

Theorem 9. *The Fourier series corresponding to $\phi(t)$ will be summable (C, n), n being a positive integer, if the Fourier series corresponding to $\phi_n(t)$ is convergent in the ordinary sense.*

Prasad's theorem I is the case $n = 1$ of the theorem. This reduces the summability (C, n) of the Fourier series for $\phi(t)$ to the ordinary convergence of the Fourier series for $\phi_n(t)$. If we now apply to $\phi_n(t)$

* Prasad, **12.**
† Verblunsky, **16.**
‡ Prasad, **12,**

the well-known standard criteria for convergence of a Fourier series, we shall get as many tests for a summability (C, n). Let us first take the case when

$$\lim_{t \to 0} \phi_n(t) = 0.$$

Then we get the following tests :—

Theorem 10. *At a point at which*

$$\lim_{t \to 0} \phi_n(t) = 0,$$

the Fourier series will be summable (C, n) *to the sum* S, n *being zero or a positive integer, provided that any one of the following conditions is satisfied :—*

(a)
$$|\varphi_n(t)| \leq A\, t^k,$$

for all values of t not greater than some fixed positive number ϵ, A *and* k *being fixed positive numbers ; or*

(b)
$$\int_0^{\epsilon} \frac{|\phi_n(t)|}{t}\, dt \quad exists\ ; or$$

(c) *an interval* $(0, \delta)$ *can be found in which* $\phi_n(t)$ *or more generally* $\dfrac{1}{t}\displaystyle\int_0^{t} \phi_n(t)\, dt$

is of bounded variation ; or

(d)
$$\int_0^{t} |\phi_{n-1}(t)|\, dt = O(t)\ ; or$$

(e)
$$\lim_{\epsilon \to 0} \int_{\epsilon}^{\delta} \left| \frac{\phi_n(t)}{t} - \frac{\phi_n(t + \epsilon)}{t + \epsilon} \right| dt = 0,\ \delta > \epsilon.$$

The conditions (a), (b), (c), (d) and (e) above are obtained by the application of convergence tests* due to Lipschitz, Dini, Jordon and de la Vallee-Poussin, Young and Lebesgue respectively. It is well known that the condition (e) includes the previous ones.

---

* G. Prasad, **13**, **25**.

If however,

$$\lim_{t \to 0} \phi_n(t)$$

does not exist but $\phi_n(t)$ has a discontinuity of the second kind at $t = 0$, then by the application of Du Bois-Reymond's test* we get the following test :—

Theorem II.   *If*

$$\phi_n(t) = \rho(t) \cos \sigma(t)$$

where each of $\rho(t)$ and $\sigma(t)$ is monotone (in $0, \delta$) and at least $\sigma(t)$ is unlimited, than the Fourier series will be summable (C, n) if

$$\log \frac{1}{t} \propto \sigma(t) \propto \left(\frac{1}{t}\right)^{\delta},$$

and

$$\rho(t) \propto t \sqrt{\sigma''(t)},$$

where δ is any positive big number.

The test given by Prasad* for summability (C, 1) are obtained from theorems 10 and 11 by putting $n = 1$.

REFERENCES

1.   L. S. Bosanquet, " On the Cesàro summability of Fourier series and its allied series," *Proc. London Math. Soc.*, (2), 37, (1934), 17-82.

2.   M. E. Grimshaw, " Summation of the integral conjugate to the Fourier integral of finite type," *Proc. Camb. Phil. Soc.*, 23, (1927), 871-881.

3.   G. H. Hardy and J. E. Littlewood, " The allied series of a Fourier series," *Proc. London Math. Soc.*, (2), 24 (1925) 211-246.

4.   ——————," On the series coujugate to the Fourier series of a bounded function," *Journal London Math. Soc.*, 6 (1931), 278-281.

5.   E. W. Hobson. *The theory of Functions of a Real Variable and the Theory of Fourier series*, Vol. 2, (1926).

6.   M. L. Misra, " A theorem on the convergence of the conjugate series of a Fourier series," *Journal London Math. Soc.*, 10, (1939), 213-216.

---

* B. N. Prasad, **12,** 44, 45.

7. —————,'' On the Poisson summability of Trigonometric series,'' *Abstracts Indian Science Congress*, 1945.

8. R. E. A. C. Paley, '' On the Cesàro summability of Fourier series and allied series,'' *Proc. Camb. Phil, Soc.*, 26, (1930), 173-203.

9. S. Pollard, '' The summation of a Fourier integral of finite type,'' *Proc. Camb. Phil. Soc.*, 23. (1927), 373-382.

10. B. N. Prasad, '' A theorem on the Cesàro summability of the allied series of a Fourier series,'' *Journal London Math. Soc.*, 6, (1931), 274-278.

11. —————, '' Contribution a l'étude de la series conjuguée,'' *Jour. de Math.*, 9 (1932), 153-205.

12. —————,'' On the summability of Fourier series by arithmetic means,' *Proc. Acad. Se. U. P. India*, 4, (1934), 39-46.

13. G. Prasad, *Recent Researches in the theory of Fourier series*, 1928.

14. A. Pringsheim, '' Ueber Das Verhalten von Potenzreiche auf dem konvergenzkreise,'' *Munchner Sitzungberichte*, 30, (1900), 37-100.

15. S. Verblunsky, '' On the limit of a function at a point,'' *Proc. London Math. Soc.*, (2), 32, (1931), 163-199.

16. —————, '' The Cesàro summation of Trignometric series,'' *Proc. London Math. Soc.*, (2), 33 (1932), 384-408.

17. W. H. Young, '' On infinite integrals involving a generalisation of the sine and cosine functions,'' *Quar. Journal Math.*, 43, (1912), 161-177.

SOME PROPERTIES OF NORMAL RECTILINEAR CONGRUENCES

By Ratan Shanker Mishra, Delhi

Rambehari[1] has shown that spherical representations of the Principal ruled surfaces through a line of a normal rectilinear congruence are isometric. In this paper I have shown that they are isothermal conjugate also and that the curves on the sphere representing the characteristic ruled surfaces through a line of a congruence formed by normals to the middle surface are also isothermal conjugate.

1. Let a rectilinear congruence be defined by

$$\xi = x + t\mathrm{X}, \quad \eta = y + t.\mathrm{Y}, \quad \zeta = z + t.\,\mathrm{Z},$$

where X, Y, Z; x, y, z are functions of two variables 'u' and 'v'.

Let the curves on the sphere which represent the principal surfaces be taken as parametric curves, then,[2]

$$\mathrm{F} = 0 \tag{1.1}$$

and

$$f + f' = 0, \tag{1.2}$$

E, F, G.; e, f, g being the coefficients of Kummer's quadratic forms.

If the congruence is normal[3], using (1.2) we get

$$f = f' = 0 \tag{1.3}$$

Taking the surface of reference as the middle surface, we get[4]

$$e\mathrm{G} + g\mathrm{E} = 0 \tag{1.4}$$

Also $\gamma \ (\equiv \Sigma \mathrm{X} \mathrm{x}_1)$ and $\gamma' \ (\equiv \Sigma \mathrm{X} \mathrm{x}_2)$ are given by[5]

$$e_2 - f_1 - \left\{{}^{12}_{\ 1}\right\}'e + \left\{{}^{11}_{\ 1}\right\}'f - \left\{{}^{12}_{\ 2}\right\}'f' + \left\{{}^{11}_{\ 2}\right\}''g + \mathrm{F}\gamma + \mathrm{E}\gamma' = 0, \tag{1.5}$$

$$f'_2 - g_1 - \left\{{}^{22}_{\ 1}\right\}'e + \left\{{}^{12}_{\ 1}\right\}'f - \left\{{}^{22}_{\ 2}\right\}'f' + \left\{{}^{12}_{\ 2}\right\}'g + \mathrm{G}\gamma - \mathrm{F}\gamma' = 0, \tag{1.6}$$

where the subscripts 1 and 2 denote differentiation with regard to 'u' and 'v' respectively.

Using (1.1) and (1.3) the equations (1.5) and (1.6) become

$$e_2 - \frac{E_2}{2} \cdot \frac{e}{E} - \frac{E_2}{2} \cdot \frac{g}{G} = 0, \tag{1.7}$$

$$-g_1 + \frac{G_1}{2} \cdot \frac{e}{E} + \frac{G_1}{2} \cdot \frac{g}{G} = 0. \tag{1.8}$$

With the help of equation (1.4) the equations (1.7) and (1.8) give

$$e_2 = g_1 = 0 \tag{1.9}.$$

∴ 'e' is a function of 'u' only and 'g' is a function of 'v' only.

$$\therefore \frac{e}{g} = \frac{\text{a function of 'u' only}}{\text{a function of 'v' only}}.$$

Hence $\dfrac{\partial^2}{\partial u \partial v} \log \left(\dfrac{e}{g} \right) = 0$ \qquad\qquad (1.10)

which shows that the parametric curves are isothermal conjugate.

Also $\dfrac{e}{g} = -\dfrac{E}{G}$ \qquad\qquad (1.11)

Therefore $\dfrac{\partial^2}{\partial u \partial v} \log \left(\dfrac{E}{G} \right) = 0,$ \qquad\qquad (1.12)

which with the help of (1.1) shows that the parametric curves are iso-metric.

Hence the curves on the unit sphere representing principal surfaces through a rectilinear congruence formed by normals to the middle surface are both isometric and isothermal conjugate.

2. Let the curves on the unit sphere representing the characteristic surfaces be parametric, then

$$\begin{vmatrix} e\,du + \tfrac{1}{2}(f+f')\,dv & \tfrac{1}{2}(f+f')\,du + g\,dv \\ \delta\,du + \delta'\,dv & \delta'\,du + \delta''\,dv \end{vmatrix} = 0 \tag{2.1}$$

must be the same as $du\,dv = 0,$

where $\delta = \dfrac{1}{H}(Ef' - Fe),\ \ 2\delta' = \dfrac{1}{H}(Eg - Ge + Ff' - Ff),\ \ \delta'' = \dfrac{1}{H}(Fg - Gf).$

Hence \qquad\qquad $f + f' = 0$ \qquad\qquad (2.2)

and \qquad\qquad $\delta' = 0$ \qquad\qquad (2.3)

Also, since the congruence consists of normals to the middle surface, which is the surface of reference, using (2.2) we get

$$f = f' = 0 \qquad\qquad (2\cdot4)$$

$$\text{and } eG + gE = 0 \qquad\qquad (2\cdot5)$$

But equation (2.3) gives

$$Eg - Ge = 0 \qquad\qquad (2\cdot6)$$

Therefore from $(2\cdot5)$ and $(2\cdot6)$ we get

$$E = G = 0, \qquad\qquad (2\cdot7)$$

since the fundamental coefficients are not proportional.

Therefore the equations $(1\cdot5)$ and $(1\cdot6)$ become with the help of $(2\cdot4)$ and (2.7)

$$e_2 - \left\{ {}^{1}_{1}{}^{2} \right\}' e + \left\{ {}^{1}_{2}{}^{1} \right\}' g = 0 \qquad\qquad (2\cdot8)$$

$$-g_1 - \left\{ {}^{2}_{1}{}^{2} \right\}' e + \left\{ {}^{1}_{2}{}^{2} \right\}' g = 0 \qquad\qquad (2\cdot9)$$

$$\text{or } e_2 = 0 \text{ and } g_1 = 0,$$

i.e. ' e ' is a function of ' u ' only and ' g ' is a function of ' v ' only.

$$\therefore \frac{\partial^2}{\partial u \partial v} \log\left(\frac{e}{g}\right) = 0.$$

Hence the curves on the unit sphere representing the characteristic surfaces through a line of a rectilinear congruence formed by normals to the middle surface are isothermal conjugate.

NOTE.—As for a normal congruence the developable surfaces coincide with principal surfaces, and surfaces whose speherical representations are minimal lines coincide with characteristic surfaces, hence the results for developable surfaces are similar to those for principal surfaces and the results fcr surfaces whose spherical representations are minimal lines are similar to those for characteristic surfaces.

For his constant guidance and valuable help in the preparation of this paper I am thankful to Dr. Ram Behari.

REFERENCES

(1) 'Proceedings of the National Academy of Sciences, India' Section A, Vol. 14, Part 2 pp. 98-100, 1944.

(2) Eisenhart, Differential Geometry of Curves and Surfaces, page 397.

(3) Eisenhart, loc. cit. page 393.

(4) Eisenhart, loc. cit. page 401 (ex. 6).

(5) Eisenhart, loc. cit. page 407.

F. 10